Data Transmission When the Sampling Frequency Exceeds the Nyquist Rate

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***Abstract*—In this letter we show that if a channel with a total spectral support of W Hz is sampled at a rate $f_s > 2W$ Hz (exceeding the Nyquist condition), it is possible to find integers M, N such that only a certain subset, consisting of M members of the N integers, carries independent data which obey $M/N \leq 2W/f_s$. Not all integers M, N satisfying this inequality will work, and even for a working set of M and N, not every subset M of N integers will work. We derive the relation between M, N, f_s, the spectral support of the channel, and the subset M of the N integers.**

***Index Terms*— Non-uniform sampling, PCM modems, 56K modems.**

I. Introduction

In this letter we are interested in data transmission over a channel band limited to W Hz, sampled at a period of T_s seconds, but where the usual Nyquist condition is violated, i.e., the sampling frequency $f_s = 1/T_s > 2W$. It is known that it is possible to transmit information within a bandwidth of W Hz while employing $f_s > 2W$, as long as the actual information rate is limited to $2W$ symbols per second [1], [2]. This can be accomplished in principle by permitting only $2W$ symbols/s to be independently chosen, with the extra $f_s - 2W$ symbols/s carrying limited or no new information, or more precisely, information which is in part determined by the independent $2W$ symbols/s [2]–[4]. A simple way to construct such signals is to permit only some subset M out of every N consecutive symbols to be independently chosen, while the remaining $N - M$ become determined by the independent samples and the bandwidth restriction.

Our basic premise in this letter is that if a channel has a total of W Hz spectral support, and if it is sampled at a rate $f_s > 2W$ Hz, then it is possible to find integers M, N such that only a certain subset, consisting of M members of the N integers, carries independent data which obey

$$\frac{M}{N} \leq \frac{2W}{f_s}. \tag{1}$$

The remainder $N - M$ integers carry data dependent on the independent data as well as the channel spectrum. These integers are ignored for data transmission purposes. A key point is that not every M, N pair, satisfying (1), will work. Furthermore, even if valid M and N are found, the choice of which particular M out of N to choose as data-bearing samples is not arbitrary, and again, not every choice will work. The next section will elaborate the limits of equalizer theory for this problem and explain how to determine both permissible M and N, as well as which particular M out of N time instances are appropriate for an arbitrary, nonideal, generalized bandlimited channel and f_s.

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II. Equalizer Theory

For the configuration in Fig. 1 where we have defined $T = NT_s$, let

$$p_k(t) = h(t) * r^k(t), \qquad 1 \leq k \leq M. \tag{2}$$

The kth equalizer, $r^k(t)$ is interpreted as the receive filter (equalizer) for the kth data-bearing member of each group of N samples, and for now, we require that the M data-bearing samples $a_n^1, a_n^2, \cdots, a_n^M$ be transmitted consecutively at $nT, nT+T_s, \cdots, nT+(M-1)T_s$. For $N-M$ time instances $nT+MT_s, \cdots, nT+(N-1)T_s$, nothing is transmitted. With this notation in place, the zero-forcing requirements for the system of Fig. 1 $p_k(t)$ can be written compactly as

$$p_k(nT + (m-1)T_s) = \delta_n \delta_{k-m}, \qquad 1 \leq k, m \leq M \tag{3}$$

for all n. From (3),

$$p_k(t) \sum_{n=-\infty}^{\infty} \delta(t - nT - (m-1)T_s) = \delta_{k-m} \delta(t - (k-1)T_s). \tag{4}$$

The Fourier transform of (4) is just

$$\frac{1}{T} \sum_{n=-\infty}^{\infty} W^{(m-1)n} P_k(f - \frac{n}{T}) = \delta_{k-m} e^{-\hat{\imath} 2\pi (k-1) f T_s} \tag{5}$$

where $W^{(m-1)n} = e^{-\hat{\imath}\frac{2\pi}{N}(m-1)n}$, $\hat{\imath} = \sqrt{-1}$, $1 \leq k, m \leq M$.

Noting that $W^i = W^{i+kN}$ for all i and k, (5) can be written as

$$\frac{1}{T} \sum_{n=0}^{N-1} W^{(m-1)n} \sum_{l=-\infty}^{\infty} P_k(f - \frac{n}{T} - \frac{l}{T_s}) = \delta_{k-m} e^{-\hat{\imath} 2\pi (k-1) f T_s} \tag{6}$$

for $1 \leq k, m \leq M$.

Equation (6) relates frequency translates of the composite folded spectra of the pulse system $p_k(t)$, which are folded modulo $1/T_s$ Hz. For any particular frequency $f_0 \in (-1/2T_s, 1/2T_s)$, N translates of f_0 are formed at $f_0 - \frac{n}{T}$, $n = 0, 1, \cdots, N-1$, the nth of these being $\sum_{l=-\infty}^{\infty} P_k(f - \frac{n}{T} - \frac{l}{T_s})$

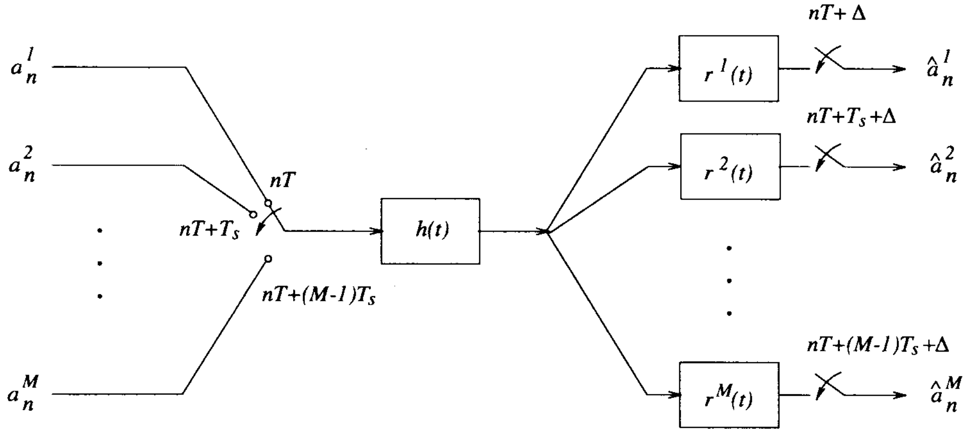

Fig. 1. Data transmission system over a channel $h(t)$ with equalizers $r^k(t)$.

which we define as $\bar{P}_k^{(n)}$. (Note the periodic nature of the folded spectrum.) Together, these $\bar{P}_k^{(n)}$ must satisfy the discrete Fourier transform relation (6), which represents (3) in the frequency domain.

Let us define $\boldsymbol{p}_k = (\bar{P}_k^{(0)}, \bar{P}_k^{(1)}, \cdots, \bar{P}_k^{(N-1)})^{\mathrm{T}}$. For notational simplicity, we will also define $\boldsymbol{W} = [W^{(i-1)(j-1)}]_{M\times N}$ and $\boldsymbol{e}_k$ as the unit vector in the kth direction. Then (6) can be written as

$$\frac{1}{T}\boldsymbol{W}\boldsymbol{p}_k = e^{-\hat{\imath}2\pi(k-1)fT_s}\boldsymbol{e}_k, \qquad 1 \le k \le M. \qquad (7)$$

The sample-rate-folded composite spectrum

$$\bar{P}_k(f) = \sum_{l=-\infty}^{\infty} P_k(f - \frac{l}{T_s}) = \sum_{l=-\infty}^{\infty} H(f - \frac{l}{T_s})R_k(f - \frac{l}{T_s}) \qquad (8)$$

may be constrained by the support of $H(f)$ to have zeros, resulting in elements of $\boldsymbol{p}_k$ which are zero. Let us define $\tilde{\boldsymbol{p}}_k$ as the vector obtained from $\boldsymbol{p}_k$ by deleting the zeros, and $\tilde{\boldsymbol{W}}$ as the matrix obtained by deleting the corresponding columns of $\boldsymbol{W}$. Then (7) becomes

$$\frac{1}{T}\tilde{\boldsymbol{W}}\tilde{\boldsymbol{p}}_k = e^{-\hat{\imath}2\pi(k-1)fT_s}\boldsymbol{e}_k, \qquad 1 \le k \le M. \qquad (9)$$

The matrix $\boldsymbol{W}$ is Fourier [5] and therefore, $\operatorname{rank}\tilde{\boldsymbol{W}} = \min(M, \dim\tilde{\boldsymbol{p}}_k)$. If $\dim\tilde{\boldsymbol{p}}_k < M$, then (9) becomes an inconsistent set of linear equations. For (9) to have *at least* one solution, we must have $\dim\tilde{\boldsymbol{p}}_k \ge M$ for all f. In other words, the configuration in Fig. 1 can support M consecutive data-bearing sample times, as described above, as long as for any $f_0 \in (-1/2T_s, 1/2T_s)$, at least M Nyquist translates lie in the support of one period of the sample-rate-folded channel spectrum $\sum_{l=-\infty}^{\infty} H(f_0 - l/T_s)$.

This requirement can be applied to the idealized passband channel with a one-sided bandwidth of 3 kHz (500–3500 Hz). Using a sampling rate $1/T_s$ of 8 kHz and $M = 6$, $N = 8$, for any choice of f_0, six 1-kHz translates are always available in the support of one period of the sample-rate folded spectrum (Fig. 2). For this channel, we assert that $M = 3$, $N = 4$ will not work. This can be demonstrated by examining the 2-kHz translations of, for example, $f_0 = 0$ Hz.

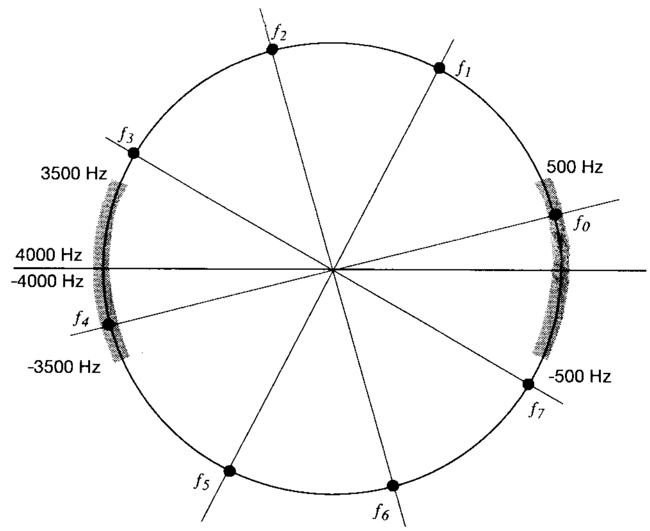

Fig. 2. Description of the spectral condition on the unit circle. For each f, there should be M nonzero Nyquist translates of the function $\bar{H}(f) = \sum_{l=-\infty}^{\infty} H(f - l/T_s)$ spaced $2\pi/N$ radians apart. Equivalently, $\bar{H}(f)$ should have M translates at $1/T$ Hz available.

In the development above, we assumed consecutive sampling times $1 \le m \le M$. When sampling is not consecutive, the analysis holds for (3)–(6) with the condition $1 \le m \le M$ replaced by the condition that m takes on M values from the set $\{1, 2, \cdots, N\}$. Let us denote this set of m's $\mathcal{M} \triangleq \{m_i : 1 \le m_i \le N \text{ for } 1 \le i \le M, m_i \ne m_j \text{ for } i \ne j\}$. Then, the matrix on the left-hand side of (6), $\boldsymbol{W} = [W^{(i-1)(j-1)}]_{M\times N}$ is replaced by $\boldsymbol{W} = [W^{(m_i-1)(j-1)}]_{M\times N}$, and the matrix on the right-hand side of (6), $[\delta_{i-j}e^{-\hat{\imath}2\pi(i-1)fT_s}]_{M\times M}$ is replaced with $[\delta_{i-j}e^{-\hat{\imath}2\pi(m_i-1)fT_s}]_{M\times M}$. Thus in this case, the nonsingularity of $M \times M$ submatrices of $\boldsymbol{W} = [W^{(m_i-1)(j-1)}]_{M\times N}$ becomes a condition for realizability, *in addition* to the spectral condition mentioned above. For example, for $\mathcal{M} = \{1, 2, 3, 5, 6, 7\}$ in the $M = 6$, $N = 8$, $1/T = 1$ kHz passband problem above, the submatrix corresponding to columns 1–3 and 5–7 of $\boldsymbol{W}$ becomes singular at, for example, $f_0 = 0$ Hz. It is known that when N is prime, all such submatrices are nonsingular [6]. Thus, when N is prime, any set of M sampling times will do, as long as, for all f, at least M nonzero

$1/T$-spaced translates are available in the sample-rate folded spectrum.

There is an interesting duality between the consecutivity of the sample times and the contiguity[1] of the sample-rate folded channel support: 1) any M sampling times can be used provided that the sample-rate folded support is contiguous and at least M translates are available, since in this case $\tilde{\boldsymbol{W}}$ is row-wise Fourier, and hence full rank and 2) any sample-rate folded support set having M translates available can be used provided that the M sampling times are consecutive, since in this case $\tilde{\boldsymbol{W}}$ is column-wise Fourier.

When $\dim \tilde{\boldsymbol{p}}_k = M$, the solution to (9) is unique. In addition, when there is only one nonzero term in each $\bar{P}_k^{(n)}$, then the equalizer set $r^k(t)$ is also unique. When $\dim \tilde{\boldsymbol{p}}_k > M$, then the system (9) is underdetermined, and an infinite set of solutions exists, though among these, there is a unique minimum-norm solution given by the pseudoinverse [7]. For the special case $\dim \tilde{\boldsymbol{p}}_k = N$, the minimum norm solution to (7) is given by the right inverse

$$\boldsymbol{p}_k = \boldsymbol{W}^H(\boldsymbol{W}\boldsymbol{W}^H)^{-1} e^{-\hat{\imath}2\pi(k-1)fT_s}\boldsymbol{e}_k, \qquad 1 \le k \le M \tag{10}$$

but since $\boldsymbol{W}\boldsymbol{W}^H = N\boldsymbol{I}$, $\boldsymbol{p}_k = e^{-\hat{\imath}2\pi(k-1)fT_s}(1, W^{-(k-1)}, W^{-2(k-1)}, \cdots, W^{-(M-1)(k-1)})^{\mathrm{T}}$.

Finally, we remark that the results in this letter are applicable to the dual of the case in Fig. 1, where the equalizers are placed at the transmitter side, preceding the channel.

Applications of this result to voiceband data transmission over a twisted pair will be described elsewhere.

[1] Contiguity here means that the support set is fully connected when depicted on the unit circle.

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