Linear Precoding Gain for Large MIMO Configurations with QAM and Reduced Complexity

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***Abstract*—In this paper, the problem of designing a linear precoder for Multiple-Input Multiple-Output (MIMO) systems in conjunction with Quadrature Amplitude Modulation (QAM) is addressed. First, a novel and efficient methodology to evaluate the input-output mutual information for a general Multiple-Input Multiple-Output (MIMO) system as well as its corresponding gradients is presented, based on the Gauss-Hermite quadrature rule. Then, the method is exploited in a block coordinate gradient ascent optimization process to determine the globally optimal linear precoder with respect to the MIMO input-output mutual information for QAM systems with relatively moderate MIMO channel sizes. The proposed methodology is next applied in conjunction with the complexity-reducing per-group processing (PGP) technique to both perfect channel state information at the transmitter (CSIT) as well as statistical channel state information (SCSI) scenarios, with large transmitting and receiving antenna size, and for constellation size up to $M = 64$. We show by numerical results that the precoders developed offer significantly better performance than the configuration with no precoder as well as the maximum diversity precoder for QAM with constellation sizes $M = 16,\ 32$, and $\ 64$ and for MIMO channel size up to 100×100.**

I. INTRODUCTION

The concept of Multiple-Input Multiple-Output (MIMO) systems still represents a prevailing research direction in wireless communications due to its ever increasing capability to offer higher rate, more efficient communications, as measured by spectral utilization, and under low transmitting or receiving power. Within MIMO research, the problem of designing an optimal linear precoder toward maximizing the mutual information between the input and output has been extensively considered. For example, in [1], [2] the optimal power allocation strategies are presented (e.g., Mercury Waterfilling (MWF)), together with general equations for the optimal precoder design. In addition, [3] also considered precoders for mutual information maximization and showed that the left eigenvectors of the optimal precoder can be set equal to the right eigenvectors of the channel.

Recently, globally optimal linear precoding techniques were presented in [4], [5] for scenarios employing perfect channel state information available at the transmitter (CSIT)[1] with finite alphabet inputs, capable of achieving mutual information rates much higher than the previously presented MWF [1] techniques by introducing input symbol correlation through a unitary input transformation matrix in conjunction with channel weight adjustment (power allocation). In addition, more recently, [6] has presented an iterative algorithm for precoder optimization for sum rate maximization of Multiple Access Channels (MAC) with Kronecker MIMO channels. Furthermore, more recent work has shown that when only statistical channel state information (SCSI)[2] is available at the transmitter, in asymptotic conditions when the number of transmitting and receiving antennas grows large, but with a constant transmitting to receiving antenna number ratio, one can design the optimal precoder by looking at an equivalent constant channel and its corresponding adjustments as per the pertinent theory [9], and applying a modified expression for the corresponding ergodic mutual information evaluation over all channel realizations. This development allows for a precoder optimization under SCSI in a much easier way [9]. However, existing research in the area does not provide any results of optimal linear precoders in the case of QAM with constellation size $M \geq 16$, with the exception of [10]. In past research work, a major impediment toward developing optimal precoders for QAM has been a lack of an accurate and efficient technique toward input-output mutual information evaluation and its gradients.

Traditionally, MIMO linear precoding has been studied in conjunction with either a) Gaussian, or b) finite alphabet inputs. The latter are more realistic and also present special issues, e.g., their offered capacity reaches saturation at high signal-to-noise ratio (SNR) [4]. Futhermore, QAM modulation with $M \geq 16$ is also instrumental in achieving higher data rates. In addition, as MIMO systems of larger transmitting and receiving sizes are becoming popular for 5G [11], linear precoding methods capable of offering high gains with low complexity for large MIMO transmitting and receiving antenna sizes and QAM alphabet size $M \geq 16$ prevail as necessary. This problem remains open, although its significance is high toward practical applications. In this paper, we propose optimal linear precoding techniques for MIMO, suitable for QAM with constellation size $M \geq 16$ which solve the above open problem. The proposed techniques can accommodate MIMO configurations with very large antenna sizes, e.g., 100×100 while their complexity grows linearly with the antenna size. The only related work in this area is [10] which has antenna sizes up to 32×32. Our approach entails a novel application of the Hermite-Gauss quadrature rule [12] which offers a very

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[1]Under CSIT the transmitter has perfect knowledge of the MIMO channel realization at each transmission.

[2]SCSI pertains to the case in which the transmitter has knowledge of only the MIMO channel correlation matrices [7], [8] and the thermal noise variance.

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accurate and efficient way to evaluate the capacity of a MIMO system with QAM. We then apply this technique within the context of a block gradient ascent method [13] in order to determine the globally optimal linear precoder for MIMO systems, in a similar fashion to [4], for systems with CSIT and small antenna size. We show that for $M = 16,\ 32$, and 64 QAM, the optimal linear precoder offers 50% better mutual information than the maximal diversity precoder (MDP) of [14] and the no-precoder case, at low signal-to-noise ratio (SNR) for a standard 2×2 MIMO channel, however the absolute utilization gain achieved is lower than $1\ b/s/Hz$. We then proceed to show that significantly higher gains are available for different channels, e.g., a utilization gain of $1.30\ b/s/Hz$ at $SNR = 10\ dB$, when $M = 16$. We then employ larger antenna configurations, e.g., up to 40×40 with CSIT and $M = 16,\ 32$, and 64 together with the complexity reducing technique of per-group processing (PGP) which was originally presented in [15], and show very high gains available with reduced system complexity. Finally, we also employ SCSI scenarios in conjunction with PGP and show very significant gains for large antenna sizes, e.g., 100×100 and $M = 16,\ 32,\ 64$. The main advantages of our work compared with other interesting proposals for large MIMO sizes, e.g., [10], lie over four main directions: a) It offers a globally optimal precoder solution for each subgroup, instead of a locally optimal one, b) It is faster, c) It allows for larger constellation size, e.g., $M = 32,\ 64$, and d) It allows larger MIMO configurations, e.g., 100×100.

The paper is organized as follows: Section II presents the system model and problem statement. Then, in Section III, we present a novel Gauss-Hermite approximation to the evaluation of the input-output mutual information of a MIMO system that allows for fast, but otherwise very accurate evaluation of the input-output mutual information of a MIMO system, and thus represents a major facilitator toward determining the globally optimal linear precoder for MIMO. In Section IV, we present numerical results for the globally optimal precoder that implements the Gauss-Hermite approximation in the block coordinate gradient ascent method. Finally, our conclusions are presented in Section V.

II. System Model and Problem Statement

The instantaneous N_t transmit antenna, N_r receive antenna MIMO model is described by the following equation

$$\mathbf{y} = \mathbf{H}\mathbf{G}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{y}$ is the $N_r \times 1$ received vector, $\mathbf{H}$ is the $N_r \times N_t$ MIMO channel matrix, $\mathbf{G}$ is the precoder matrix of size $N_t \times N_t$, $\mathbf{x}$ is the $N_t \times 1$ data vector with independent components each of which is in the QAM constellation of size M, $\mathbf{n}$ represents the circularly symmetric complex Additive White Gaussian Noise (AWGN) of size $N_r \times 1$, with mean zero and covariance matrix $\mathbf{K}_n = \sigma_n^2 \mathbf{I}_{N_r}$, where $\mathbf{I}_{N_r}$ is the $N_r \times N_r$ identity matrix, and $\sigma^2 = \frac{1}{SNR}$, SNR being the (coded) symbol signal-to-noise ratio. In this paper, a number of different channels will be considered, e.g., channels comprising independent complex Gaussian components or spatially correlated Kronecker-type channels [7] (including those similar to the 3GPP spatial correlation model (SCM) [16]), or more generally Weichselberger channels [8]. The precoding matrix $\mathbf{G}$ needs to satisfy the following power constraint

$$\mathrm{tr}(\mathbf{G}\mathbf{G}^h) = N_t, \tag{2}$$

where $\mathrm{tr}(\mathbf{A})$, $\mathbf{A}^h$ denote the trace and the Hermitian transpose of matrix $\mathbf{A}$, respectively. An equivalent model called herein the "virtual" channel is given by [4]

$$\mathbf{y} = \mathbf{\Sigma}_H \mathbf{\Sigma}_G \mathbf{V}_G^h \mathbf{x} + \mathbf{n}, \tag{3}$$

where $\mathbf{\Sigma}_H$ and $\mathbf{\Sigma}_G$ are diagonal matrices containing the singular values of $\mathbf{H}$, $\mathbf{G}$, respectively and $\mathbf{V}_G$ is the matrix of the right singular vectors of $\mathbf{G}$. When a capacity-approaching Forward Error Correction (FEC) code is employed in this MIMO system, the overall utilization in $b/s/Hz$ is determined by the mutual information between the transmitting branches $\mathbf{x}$ and the receiving ones, $\mathbf{y}$ [17], [18]. It is shown [4] that the mutual information between $\mathbf{x}$ and $\mathbf{y}$, for channel realization $\mathbf{H}$, $I(\mathbf{x};\mathbf{y})$, is only a function of $\mathbf{W} = \mathbf{V}_G \mathbf{\Sigma}_H^2 \mathbf{\Sigma}_G^2 \mathbf{V}_G^h$. The optimal CSIT precoder $\mathbf{G}$ is found by solving:

$$\begin{aligned} \underset{\boldsymbol{G}}{\text{maximize}} \quad & I(\mathbf{x};\mathbf{y}) \\ \text{subject to} \quad & \mathrm{tr}(\mathbf{G}\mathbf{G}^h) = N_t, \end{aligned} \tag{4}$$

called the "original problem," and

$$\begin{aligned} \underset{\boldsymbol{V}_G, \mathbf{\Sigma}_G}{\text{maximize}} \quad & I(\mathbf{x};\mathbf{y}) \\ \text{subject to} \quad & \mathrm{tr}(\mathbf{\Sigma}_G^2) = N_t, \end{aligned} \tag{5}$$

called the "equivalent problem," where the reception model of (3) is employed. The solution to (4) or (5) results in exponential complexity at both transmitter and receiver, and it becomes especially difficult for QAM with constellation size $M \geq 16$ or large MIMO configurations. A major difficulty in the QAM case stems from the fact that there are multiple evaluations of $I(\mathbf{x};\mathbf{y})$ in the block coordinate ascent method employed for determining the globally optimal precoder. More specifically, for each block coordinate gradient ascent iteration, there are two line backtracking searches required [4], which demand one $I(\mathbf{x};\mathbf{y})$ plus its gradient evaluations per search trial, and one additional evaluation at the end of a successful search per backtracking line search. Thus, the need of a fast, but otherwise very accurate method of calculating $I(\mathbf{x};\mathbf{y})$ and its gradients prevails as instrumental toward determining the globally optimal linear precoder for CSIT. In the SCSI case, the corresponding optimization problem becomes

$$\begin{aligned} \underset{\boldsymbol{G}}{\text{maximize}} \quad & \mathbb{E}_{\mathbf{H}}\left\{I(\mathbf{x};\mathbf{y})\right\} \\ \text{subject to} \quad & \mathrm{tr}(\mathbf{G}\mathbf{G}^h) = N_t, \end{aligned} \tag{6}$$

where the expectation is performed over all the channels $\mathbf{H}$. The ground-breaking work of [9] has shown that the problem in (6) for large antenna sizes can be solved by an approximate way of calculating the ergodic mutual information $\mathbb{E}_{\mathbf{H}}\left\{I(\mathbf{x};\mathbf{y})\right\}$ for a fixed precoding matrix $\mathbf{G}$ through well-determined parameters of a deterministic channel, including the mutual information of the corresponding deterministic channel, i.e., a CSIT scenario. Thus, methods that offer simplification of CSIT mutual information evaluation, $I(\mathbf{x};\mathbf{y})$, are also important in the SCSI case toward determining the globally optimal linear precoder for MIMO.

III. Accurate Approximation to $I(\mathbf{x};\mathbf{y})$ for MIMO Systems Based on Gauss-Hermite Quadrature

In Appendix A we prove that by applying the Gauss-Hermite quadrature theory for approximating the integral of a Gaussian function multiplied with an arbitrary real function $f(x)$, i.e.,

$$F \doteq \int_{-\infty}^{+\infty} \exp(-x^2) f(x) dx, \tag{7}$$

which is approximated in the Gauss-Hermite approximation with L weights and nodes as

$$F \approx \sum_{l=1}^{L} c(l) f(v_l) = \mathbf{c}^t \mathbf{f}, \tag{8}$$

with $\mathbf{c} = [c(1) \cdots c(L)]^t$, $\{v_l\}_{l=1}^{L}$, and $\mathbf{f} = [f(v_1) \cdots f(v_L)]^t$, being the vector of the weights, the nodes, and function node values, respectively (see Appendix A), a very accurate approximation is derived for $I(\mathbf{x};\mathbf{y})$ in a MIMO system, as presented in the following lemma. Let us first introduce some notations that make the overall understanding easier. Let $\mathbf{n}_e$ denote the equivalent to $\mathbf{n}$, real vector of length $2N_r$ derived from $\mathbf{n}$ by separating its real and imaginary parts as $\mathbf{n}_e = [n_{r1}\ n_{i1} \cdots n_{rN_r}\ n_{iN_r}]^t$, with $n_{rv},\ n_{iv}$ being the values of the real, imaginary part of the vth ($1 \le v \le N_r$) element of $\mathbf{n}$, respectively. Let us also define the real vector $\mathbf{v}(\{k_{rv},\ k_{iv}\}_{v=1}^{N_r})$ of length $2N_r$ defined as follows

$$\mathbf{v}(\{k_{rv},\ k_{iv}\}_{v=1}^{N_r}) = [v_{kr1}\ v_{ki1}, \cdots, v_{krN_r}\ v_{kiN_r}]^t, \tag{9}$$

with $k_{rv},\ k_{iv}$ ($1 \le v \le N_r$) being permutations of indexes in the set $\{1, 2, \cdots, L\}$. Then the following lemma is true concerning the Gauss-Hermite approximation for $I(\mathbf{x};\mathbf{y})$.

Lemma 1. *For the MIMO channel model presented in (1), the Gauss-Hermite approximation for $I(\mathbf{x};\mathbf{y})$ with L nodes per receiving antenna is given as*

$$I(\mathbf{x};\mathbf{y}) \approx N_t \log_2(M) - \frac{N_r}{\log(2)} - \frac{1}{M^{N_t}} \sum_{k=1}^{M^{N_t}} \hat{f}_k, \tag{10}$$

where

$$\hat{f}_k = \left(\frac{1}{\pi}\right)^{N_r} \sum_{k_{r1}=1}^{L} \sum_{k_{i1}=1}^{L} \cdots \sum_{k_{rN_r}=1}^{L} \sum_{k_{iN_r}=1}^{L} c(k_{r1}) c(k_{i1}) \cdots$$
$$c(k_{rN_r}) c(k_{iN_r}) g_k(\sigma n_{k_{r1}}, \sigma n_{k_{i1}}, \cdots, \sigma n_{k_{rN_r}}, \sigma n_{k_{iN_r}}), \tag{11}$$

with

$$g_k(\sigma v_{k_{r1}}, \sigma v_{k_{i1}}, \cdots, \sigma v_{k_{rN_r}}, \sigma v_{k_{iN_r}}) \tag{12}$$

being the value of the function

$$\log_2 \left(\sum_m \exp(-\frac{1}{\sigma^2} ||\mathbf{n} - \mathbf{H}\mathbf{G}(\mathbf{x}_k - \mathbf{x}_m)||^2) \right) \tag{13}$$

evaluated at $\mathbf{n}_e = \sigma \mathbf{v}(\{k_{rv},\ k_{iv}\}_{v=1}^{N_r})$.

The proof of this lemma is presented in Appendix A, together with a simplification available for this expression in the $N_t = N_r = 2$ case. Let us stress that, the presented novel application of the Gauss-Hermite quadrature in the MIMO model allows for efficient evaluation of $I(\mathbf{x};\mathbf{y})$ for any channel matrix $\mathbf{H}$, and precoder $\mathbf{G}$, as required in the precoder optimization process, as explained below.

IV. Global Optimization over $\mathbf{G}$ toward Maximum $I(\mathbf{x};\mathbf{y})$ for QAM

A. Description of the Globally Optimal Precoder Method

Similarly to [4], we follow a block coordinate gradient ascent maximization method to find the solution to the optimization problem described in (4), employing the virtual model of (3). It is proven in [4] that $I(\mathbf{x};\mathbf{y})$ is a concave function over $\mathbf{W}$ and $\mathbf{\Sigma}_G^2$. It thus becomes efficient to employ two different gradient ascent methods, one for $\mathbf{W}$, and another one for $\mathbf{\Sigma}_G^2$. We employ $\mathbf{\Theta}$ and $\mathbf{\Sigma}$ to denote $\mathbf{V}_G^h$ and $\mathbf{\Sigma}_G^2$, respectively, evaluated during the execution of the optimization algorithm.The algorithm's pseudocode for a number of iterations t is presented under the heading Algorithm 1.

Algorithm 1 Global precoder optimization algorithm with t iterations

procedure Precoder($\mathbf{\Sigma}_H$)
  while $i \le t$ **do**
    Determine $\mathbf{W}_{NEW} = \mathbf{\Theta}^h \mathbf{\Sigma}^2 \mathbf{\Theta}$ through backtracking line search
    Set $\mathbf{V}_G^h = \mathbf{\Theta}$
    Determine $\mathbf{W}_{NEW} = \mathbf{V}_G \mathbf{\Sigma}_G^2 \mathbf{\Sigma}_H^2 \mathbf{V}_G^h$
    Determine $\mathbf{\Sigma}_{NEW,G}^2$ through backtracking line search
    Set negative entries on the diagonal of $\mathbf{\Sigma}_{NEW,G}^2$ to zero
    Normalize $\mathbf{\Sigma}_{NEW,G}^2$ to a trace equal to N_t
    Set $\mathbf{\Sigma}_G = \mathbf{\Sigma}_{NEW,G}$
    Determine $\mathbf{W}_{NEW} = \mathbf{V}_G \mathbf{\Sigma}_G^2 \mathbf{\Sigma}_H^2 \mathbf{V}_G^h$
    Set $\mathbf{W} = \mathbf{W}_{NEW}$
    Evaluate $I(\mathbf{W})$
  end while
  return $I(\mathbf{W})$
end procedure

B. Determination of $\nabla_{\mathbf{W}} I,\ \nabla_{\mathbf{\Sigma}_G^2} I$

We first set $\mathbf{M} = \mathbf{W}^{\frac{1}{2}}$. Then, it is easy to see that I is a function of $\mathbf{M}$ (see, e.g., [4] where the notion of sufficient statistic is employed to show that $I(\mathbf{x};\mathbf{y})$ depends on $\mathbf{W}$). The derivation of $\nabla_{\mathbf{W}} I$ is presented in Appendix B. The proof is based on the following theorem[3].

Theorem 1. *Substituting $\mathbf{M} = \mathbf{V}_G \mathbf{\Sigma}_H \mathbf{\Sigma}_G \mathbf{V}_G^h = \mathbf{W}^{\frac{1}{2}}$ for $\mathbf{H}\mathbf{G}$ in (10) results in the same value of $I(\mathbf{x};\mathbf{y})$. In other*

[3]The theorem applies without loss of generality to the $N_t = N_r$ case. If $N_t \neq N_r$, then $\mathbf{\Sigma}_H$ needs to be either shrunk, or extended in size, by elimination or addition of zeros, respectively.

words, since $\mathbf{M}$ *is a function of* $\mathbf{H}$, $\mathbf{G}$*,* $\mathbf{My}$ *is a sufficient statistic for* $\mathbf{y}$.

Proof. The proof of the theorem is simple. First, recall that the "virtual" channel model in (3) is equivalent to the following model, which results by multiplying (3) by the unitary matrix $\mathbf{V}_G$ on the left, resulting in

$$\tilde{\mathbf{y}} = \mathbf{V}_G \mathbf{y} = \mathbf{V}_G \mathbf{\Sigma}_H \mathbf{\Sigma}_G \mathbf{V}_G^h \mathbf{x} + \mathbf{V}_G \mathbf{n}, \tag{14}$$

where the modified noise term $\mathbf{V}_G\mathbf{n}$ has the same statistics with $\mathbf{n}$, because $\mathbf{V}_G$ is unitary. By applying the Gauss-Hermite approximation to (14), we see that we get the desired result, i.e., the value of $I(\mathbf{x};\mathbf{y})$ remains the same, since both channel manifestations represent equivalent channels, i.e., the original one and its equivalent, thus their mutual information is the same. This completes the proof of the theorem. □

Assume without loss of generality that $N_t = N_r$. The gradient of I with respect to $\mathbf{M}$ can be found (see Appendix B for the derivation) from the Gauss-Hermite expression presented in (10) as follows

$$\nabla_{\mathbf{M}} I = -\frac{1}{\log(2)} \frac{1}{M^{N_t}} \left(\frac{1}{\pi}\right)^{N_r} \sum_{k_{r1}=1}^{L} \sum_{k_{i1}=1}^{L} \cdots \sum_{k_{rN_r}=1}^{L} \sum_{k_{iN_r}=1}^{L} c(k_{r1}) \cdots c(k_{iN_r}) \mathbf{R}(\sigma v_{k_{r1}}, \sigma v_{k_{i1}}, \cdots, \sigma v_{k_{rN_r}}, \sigma v_{k_{iN_r}}), \tag{15}$$

where $\mathbf{R}(\sigma v_{k_{r1}}, \sigma v_{k_{i1}}, \cdots, \sigma v_{k_{rN_r}}, \sigma v_{k_{iN_r}})$ is the value of the $N_t \times N_t$ matrix

$$\begin{aligned} &\sum_k \frac{1}{\sum_m \exp(-\frac{1}{\sigma^2}||\mathbf{n} - \mathbf{M}(\mathbf{x}_k - \mathbf{x}_m)||^2)} \\ &\times \sum_m \exp(-\frac{1}{\sigma^2}||\mathbf{n} - \mathbf{M}(\mathbf{x}_k - \mathbf{x}_m)||^2) \\ &\times ((\mathbf{n} - \mathbf{M}(\mathbf{x}_k - \mathbf{x}_m))(\mathbf{x}_k - \mathbf{x}_m)^h \\ &+ ((\mathbf{n} - \mathbf{M}(\mathbf{x}_k - \mathbf{x}_m))(\mathbf{x}_k - \mathbf{x}_m)^h)^h) \end{aligned} \tag{16}$$

evaluated at $\mathbf{n}_e = \sigma \mathbf{v}(\{k_{rv},\ k_{iv}\}_{v=1}^{N_r})$.

The required $\nabla_{\mathbf{W}} I$ for the execution of the optimization process can be found from Appendix B as per the next lemma, using an easily proven equation. Using the fact that for a Hermitian matrix such as $\mathbf{M}$, we need to add the Hermitian of the differential above in order to evaluate the actual gradient (see [19]), we get the desired result as follows (see Appendix B).

Lemma 2. *For the MIMO channel model presented in (1), the Gauss-Hermite approximation allows to approximate* $\nabla_{\mathbf{W}} I$ *as follows.*

$$\nabla_{\mathbf{W}} I \approx \mathrm{reshape}((\mathrm{vec}(\nabla_{\mathbf{M}} I)^T ((\mathbf{M}^*) \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{M})^{-1}), N_t, N_t), \tag{17}$$

where $\mathrm{reshape}(\mathbf{A}, k, n)$ *is the standard reshape of a matrix* $\mathbf{A}$ *(with total number of elements* kn*) to a matrix with* k *rows,* n *columns, and where* $\otimes$ *denotes Kronecker product of matrices.*

Standard reshape emanates from the vector $\mathrm{vec}(\mathbf{A})$ of matrix $\mathbf{A}$ which encompasses all columns of $\mathbf{A}$ starting from the leftmost one to the rightmost. Then, as $I(\mathbf{x};\mathbf{y})$ is a concave function of $\mathbf{W}$ [4], we can maximize over $\mathbf{W}$ in a straightforward way using closed form expressions. This is based on the fact that the approximated $I(\mathbf{x};\mathbf{y})$ through the Gauss-Hermite approximation is very accurate, as shown in the next section.

V. NUMERICAL RESULTS

The results presented in this subsection employ QAM with 16, 32, or 64 constellation sizes. We employ MIMO systems with $N_t = N_r = 2$ when global precoding optimization is performed. We have used an $L = 3$ Gauss-Hermite approximation which results in 3^{2N_r} total nodes due to MIMO. The implementation of the globally optimizing methodology is performed by employing two backtracking line searches, one for $\mathbf{W}$ and another one for $\mathbf{\Sigma}_G^2$ at each iteration, in a fashion similar to [4]. For the results presented, it is worth mentioning that only a few iterations (e.g., typically < 8) are required to converge to the optimal solution results as presented in this paper. We apply the complexity reducing method of PGP [15] which offers semi-optimal results under exponentially lower transmitter and receiver complexity [15]. For all cases presented here, we use PGP with a group size of 2×2. Based on [15] this method achieves a complexity reduction at both the transmitter precoding design and the receiver maximum a posteriori (MAP) detector on the order of $\frac{2}{N_t} M^{2(N_t-2)}$.

We divide this section into four subsections. In the first subsection, we examine the accuracy of the proposed Gauss-Hermite approximation and provide a comparison with the lower bound technique presented in [20]. In the second subsection we present results for SCSI channels similar to the ones in 3GPP SCM [16], with antenna size up to 100×100 with PGP and modulation size $M = 16$. In the third subsection, we present results for CSIT jointly with PGP and high size of antennas and modulation. Finally, in the last subsection, we present results for a MIMO system with 100 base station antennas and 4 user antennas with $M = 16,\ 64$.

A. Accuracy of the Gauss-Hermite approximation technique

In Fig. 1 we present results for I through simulation, the Gauss-Hermite approximation (GH) with $L = 3$, and the lower bound developed in [20] versus the signal-to-noise ratio per bit (SNR_b) in dB, for QAM with $M = 16$, and for the commonly used channel [4], [14],

$$\mathbf{H}_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

In the same figure, we also show a lower bound for $I(\mathbf{x};\mathbf{y})$ which appeared in [20]. We see excellent accuracy for the approximation, i.e., no observable difference between the Gauss-Hermite approximation and the simulations, over all SNR_b values.

B. Results for SCSI in Conjunction with PGP

In Fig. 2 we present results for PGP versus a no-precoding urban SCM channel with half-wavelength antenna element spacing [16] and with $N_t = N_r = 100$ and $M = 16$.

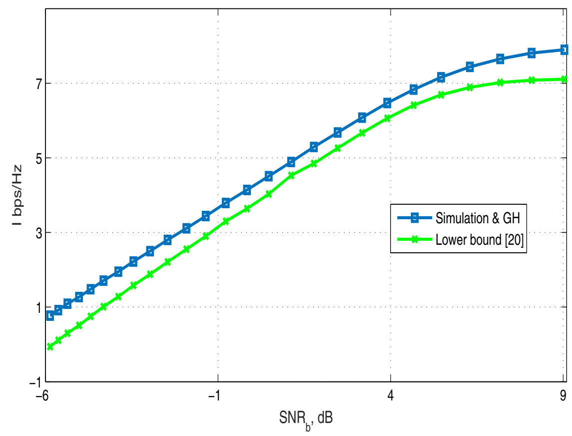

Fig. 1. Results for $I(\mathbf{x};\mathbf{y})$ without precoding for the $\mathbf{H}_1$ channel and QAM $M = 16$ modulation.

To the best of our knowledge, results for such large MIMO configurations are not available in the literature. Similar to

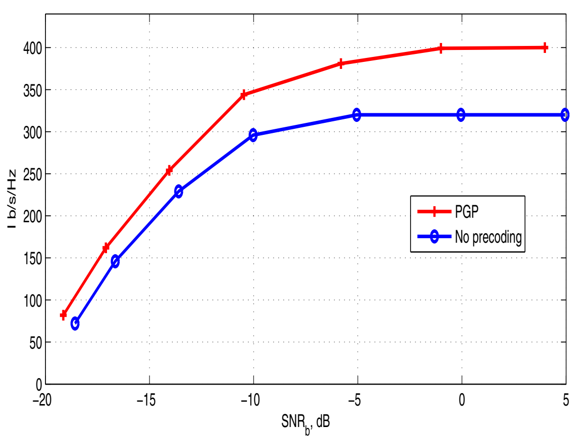

Fig. 2. $I(\mathbf{x};\mathbf{y})$ results for PGP and no-precoding cases for a 100×100 $\mathbf{H}$ SCSI MIMO system and QAM $M = 16$ modulation.

the previous results, we observe high information rate gains in the high SNR_b regime as the PGP system achieves the full capacity of 400 $b/s/Hz$ while the no-precoding scheme saturates at 320 $b/s/Hz$. The PGP system employed uses 50 groups of size 2×2 each.

C. Results for CSIT in Conjunction with PGP

In Fig. 3 we present results for an asymmetric randomly generated MIMO channel with $N_t = 4,\ N_r = 10$, and $M = 16$. PGP employs two groups of size $N_r = 5,\ N_t = 2$ each. In the current scenario, we observe that significant gains are shown in the low SNR_b regime, e.g., around 3 dB in SNR_b lower than -7 dB.

D. Results for Massive MIMO

Massive MIMO [11], [21], [22] has attracted much interest recently, due to its potential to offer high data rates. We

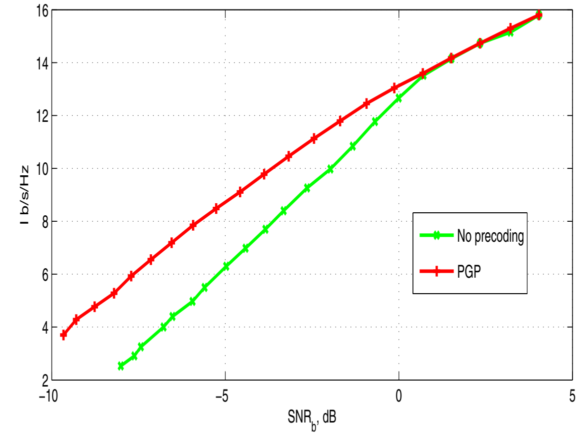

Fig. 3. $I(\mathbf{x};\mathbf{y})$ results for PGP and no-precoding cases for a randomly generated 10×4 $\mathbf{H}$ CSIT MIMO system and QAM $M = 16$ modulation.

present results for the uplink, and downlink of a Massive MIMO system based on 100 base station, 4 user antennas, respectively, with $M = 16,\ 64$, and for a Kronecker-based 3GPP SCM urban channel in a CSIT scenario in a single user configuration. Fig. 4 shows results for the 100×4 uplink of the system. We employ PGP to dramatically reduce the system complexity at the transmitter and receiver sites. Under

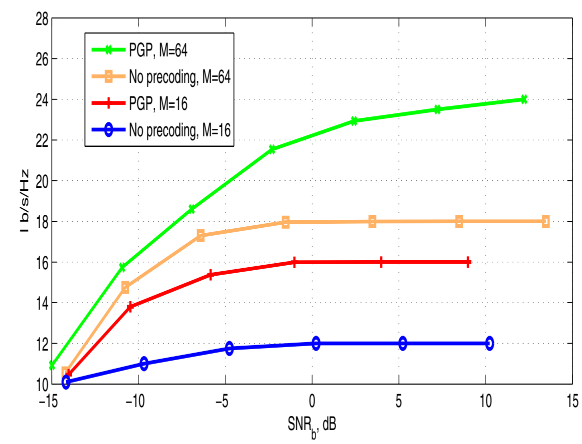

Fig. 4. $I(\mathbf{x};\mathbf{y})$ results for PGP and no-precoding cases for a randomly generated 100×4 uplink $\mathbf{H}$ CSIT MIMO system and QAM $M = 16,\ 64$ modulation.

no precoding, the channel saturates and fails to meet the maximum possible mutual information of 16 $b/s/Hz$, while with PGP the system clearly achieves the maximum mutual information rate, thus achieving high gains on the uplink in the high SNR regime. We stress the much higher throughput possible with $M = 64$ over the $M = 16$ case. For example, the no-precoding $M = 16$ uplink significantly outperforms the PGP $M = 16$ uplink. Second, the PGP $M = 64$ uplink offers further gains by, e.g., achieving the maximum possible rate of 24 $b/s/Hz$. For the downlink, in Fig. 5 we show results where the no-precoding case operates under 100 antenna inputs all correlated through the right eigenvectors of the channel, thus creating a very demanding environment at the user, due to the

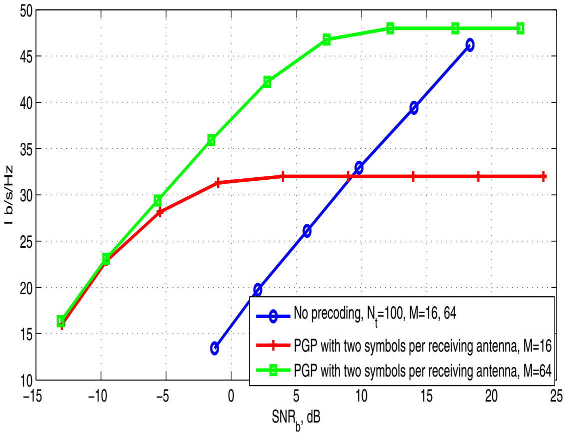

Fig. 5. $I(\mathbf{x};\mathbf{y})$ results for PGP and no-precoding cases for a randomly generated 4×100 $\mathbf{H}$ downlink CSIT MIMO system and QAM $M = 16,\ 64$ modulation.

exponentially increasing MAP detector complexity [15]. On the other hand, employing PGP with only two input symbols per receiving antenna, i.e., with dramatically reduced decoding complexity, the PGP system achieves much higher throughput in the lower SNR regime, with SNR gain on the order of $10\ dB$, albeit achieving a maximum of $32,\ 48\ b/s/Hz$ as there are a total of 8 $M = 16,\ 64$ QAM data symbols employed, respectively. We observe the superior performance of $M = 64$ over its $M = 16$ counterpart due to its increased constellation size. For example, at medium SNR_b, e.g., $SNR_b = 4$, the $M = 64$ PGP scheme achieves 45% higher throughput that the $M = 16$ one, a significant improvement. We would also like to emphasize that the no-precoding scheme requires a very high exponential MAP detector complexity, on the order of M^{200}, while for the low-SNR-superior PGP, this complexity is on the order of M^4 only. Thus, even in the higher SNR region where the no-precoding scheme can achieve a higher throughput, the complexity required at the user site becomes prohibitive. This demonstrates the superiority of PGP on the Massive MIMO downlink. On the other hand, in lower SNR, the PGP scheme achieves both much higher throughput with simultaneously exponentially lower MAP detector complexity at the user site detector. Note that the performance provided by PGP on the downlink depends on the number of symbols processed jointly. This number is currently limited by the computational complexity available. This limitation does not occur on the uplink since in that case $N_t < N_r$.

VI. Conclusions

In this paper, the problem of designing a linear precoder for MIMO systems toward mutual information maximization is addressed for QAM with $M \geq 16$ and in conjunction with large MIMO system size. A major obstacle toward this goal is a lack of efficient techniques for evaluating $I(\mathbf{x}, \mathbf{y})$ and its derivatives. We have presented a novel solution to this problem based on the Gauss-Hermite quadrature. We then applied a global optimization framework to derive the globally optimal precoder for the case of QAM with $M = 16$ and 32 and small antenna size configurations. We showed that under CSIT in this case, significant gains are available for the lower SNR range over no precoding, or MDP. We showed that for the standard 2×2 channel, although the globally optimal precoder offers significant gains over MDP and the no-precoder configurations in the low SNR region, it fails to offer gains as high as $1\ b/s/Hz$. However, we demonstrated that by employing another 2×2 channel gains as high as $1.4\ b/s/Hz$ are possible. For systems of large MIMO configurations, we applied the complexity-simplifying PGP concept [15] to derive semi-optimal precoding results. Under SCSI, we showed that for urban 3GPP SCM channels, an interesting saturation effect in the no-precoding case takes place, while the semi-optimal PGP precoder offers dramatically better results in this case while it does not experience any saturation as the SNR increases, e.g., it achieves the maximum information rate, $I(\mathbf{x};\mathbf{y}) = N_t \log_2(M)$ at high SNR. Furthermore, we applied the same Gauss-Hermite approximation approach to CSIT with a large number of antenna with the same success. We showed that for specific type of channels similar to urban 3GPP SCM [16], the PGP approach offers very high gains over the no-precoding case in the high SNR_b regime. Finally, we considered a Massive MIMO scenario in conjunction with CSIT and showed that by carefully designing the downlink and uplink precoders, the methodology shows very high gains, especially on the downlink, although it employs an exponentially simpler MAP detector at the user site.

Based on the evidence presented, the novel application of the Gauss-Hermite quadrature rule in the MIMO scenario allows for generalizing the interesting results presented in [4], [9] to the QAM case with ease. Because of the simplification achieved by the combination of PGP and the Gauss-Hermite approximation, we were able to derive results with, e.g., $N_t = N_r = 100$ as well as with $M = 64$ efficiently.

Appendix A
Gauss-Hermite Quadrature Approximation in MIMO Input Output Mutual Information

$I(\mathbf{x};\mathbf{y}) = H(\mathbf{x}) - H(\mathbf{x}|\mathbf{y}) = N_t \log_2(M) - H(\mathbf{x}|\mathbf{y})$, where the conditional entropy, $H(\mathbf{x}|\mathbf{y})$ can be written as [4]

$$\begin{aligned} &H(\mathbf{x}|\mathbf{y}) = \frac{N_r}{\log(2)} + \frac{1}{M^{N_t}} \sum_k \\ &\mathbb{E}_{\mathbf{n}} \left(\log_2 \left(\sum_m \exp(-\frac{1}{\sigma^2} ||\mathbf{n} - \mathbf{HG}(\mathbf{x}_k - \mathbf{x}_m)||^2) \right) \right) = \\ &\frac{N_r}{\log(2)} + \frac{1}{M^{N_t}} \sum_k \int_{-\infty}^{+\infty} \mathcal{N}_c(\mathbf{n}|\mathbf{0}, \sigma^2\mathbf{I}) \\ &\times \log_2 \left(\sum_m \exp(-\frac{1}{\sigma^2} ||\mathbf{n} - \mathbf{HG}(\mathbf{x}_k - \mathbf{x}_m)||^2) \right) d\mathbf{n}, \end{aligned} \tag{18}$$

where $\mathcal{N}_c(\mathbf{n}|\mathbf{0}, \sigma^2\mathbf{I})$ represents the probability density function (pdf) of the circularly symmetric complex AWGN. Let us define

$$f_k \doteq \int_{-\infty}^{+\infty} \mathcal{N}_c(\mathbf{n}|\mathbf{0}, \sigma^2\mathbf{I}) \times \log_2\left(\sum_{\mathbf{x}_m} \exp(-\frac{1}{\sigma^2}||\mathbf{n} - \mathbf{HG}(\mathbf{x}_k - \mathbf{x}_m)||^2)\right) d\mathbf{n}. \quad (19)$$

The integral above can be partitioned into $2N_r$ real integrals in tandem, in the following manner: Define by n_{rv}, n_{iv}, with $v = 1, \cdots, N_r$, the vth receiving antenna real and imaginary noise component, respectively. Also define by $(\mathbf{HG}(\mathbf{x}_k - \mathbf{x}_m))_{rv}$ and $(\mathbf{HG}(\mathbf{x}_k - \mathbf{x}_m))_{iv}$, the vth receiving antenna real and imaginary component of $(\mathbf{HG}(\mathbf{x}_k - \mathbf{x}_m))$, respectively. The Gauss-Hermite quadrature is $\int_{-\infty}^{+\infty} \exp(-x^2) f(x) dx \approx \sum_{l=1}^{L} c(l) f(v_l)$, for any real function $f(x)$, and with vector $\mathbf{c} = [c(1) \cdots c(L)]^T$ being the "weights," and v_l are the "nodes" of the approximation. The approximation is based on the following weights and nodes [12]

$$c(l) = \frac{2^{L-1} L! \sqrt{2}\pi}{L^2 (H_{L-1}(v_l))^2} \quad (20)$$

where $H_{L-1}(x) = (-1)^{L-1} \exp(x^2) \frac{d^{L-1}}{dx^{L-1}}(\exp(-x^2))$ is the $(L-1)$th order Hermitian polynomial, and the value of the node v_l equals the root of $H_L(x)$ for $l = 1, 2, \cdots, L$.

Applying the Gauss-Hermite quadrature $2N_r$ times in tandem, to the integral in (19), and after changing variables, we get that

$$f_k \approx \hat{f}_k = \left(\frac{1}{\pi}\right)^{N_r} \sum_{k_{r1}=1}^{L} \sum_{k_{i1}=1}^{L} \cdots \sum_{k_{rN_r}=1}^{L} \sum_{k_{iN_r}=1}^{L} c(k_{r1}) c(k_{i1}) \cdots \times c(k_{iN_r}) g_k(\sigma n_{k_{r1}}, \sigma n_{k_{i1}}, \cdots, \sigma n_{k_{rN_r}}, \sigma n_{k_{iN_r}}), \quad (21)$$

where $g_k(\sigma n_{k_{r1}}, \sigma n_{k_{i1}}, \cdots, \sigma n_{k_{rN_r}}, \sigma n_{k_{iN_r}})$ is the value of the function (from (21)) $\log_2\left(\sum_m \exp(-\frac{1}{\sigma^2}||\mathbf{n} - \mathbf{HG}(\mathbf{x}_k - \mathbf{x}_m)||^2)\right)$ evaluated at $\mathbf{n}_e = \sigma \mathbf{v}(\{k_{rv},\ k_{iv}\}_{v=1}^{N_r})$.

APPENDIX B
DERIVATION OF $\nabla_{\mathbf{W}} I$ THROUGH THE GAUSS-HERMITE APPROXIMATION

Without loss of generality, let's assume that $N_t = N_r$. Using Theorem 1, we can write by using the Gauss-Hermite approximation with $\mathbf{M}$ instead of $\mathbf{HG}$,

$$I(\mathbf{x}; \mathbf{y}) \approx N_t \log_2(M) - \frac{N_r}{\log(2)} - \frac{1}{M^{N_t}} \sum_k \hat{f}_k. \quad (22)$$

In order to derive the gradient of I with respect to $\mathbf{W}$, we first derive the gradient of I with respect to $\mathbf{M}$. Start with the differential of I with respect to $\mathbf{M}^*$ in (22) and approximate the f_k by $\hat{f}_k$, to get for the differential of $I(\mathbf{x}; \mathbf{y})$ over $\mathbf{M}^*$. The full details can be found in [23], but are omitted due to lack of space. We can then employ identities from [19] to get the desired result.

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