Diversity Analysis of Bit-Interleaved Coded Multiple Beamforming

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***Abstract*—In this paper, diversity analysis of bit-interleaved coded multiple beamforming (BICMB) is extended to the case of general spatial interleavers, removing a condition on their previously known design criteria. We provide a method to get diversity order, simplifying the calculation of pairwise error probability (PEP). By using the Singleton bound, we also show the maximum achievable diversity for given code rate and the number of subchannels.**

***Index Terms*—Bit-interleaved coded modulation (BICM), singular value decomposition (SVD), multi-input multi-output (MIMO), Wishart matrices, convolutional codes.**

I. Introduction

WHEN the channel information is perfectly available at the transmitter, beamforming is an attractive technique to enhance the performance of a multi-input multi-output (MIMO) system [1]. The beamforming matrices can be obtained by singular value decomposition (SVD) which is optimal in terms of minimizing the average bit error rate (BER) [2]. Single beamforming, which carries only one symbol at a time, was shown to achieve the full diversity order of NM where N is the number of transmit antennas and M is the number of receive antennas [3], [4]. However, multiple beamforming, which increases the throughput by sending multiple symbols at a time, loses the full diversity order over flat fading channels. To achieve the full diversity order in multiple beamforming, the authors in [5] introduced BICMB, combining bit-interleaved coded modulation (BICM) and multiple beamforming. Design criteria for interleaving the coded sequence were provided [5], [6].

In this paper, we demonstrate a method to determine whether a given BICMB system satisfies the design criteria by adapting Viterbi's transfer function as in finding weight spectra of the convolutional code. A similar method appears in the literature on the performance analysis of single-input single-output (SISO) coded systems over block fading channels [7], [8]. However, we apply Viterbi's approach to the bit-interleaved coded MIMO systems. Through this method, we verify the previous known result that the BICMB system with the $1/2$-rate convolutional encoder, a simple interleaver, and the soft-input Viterbi decoder achieves the full diversity order when it is used in a 2×2 system with 2 streams. In addition, we will present diversity analysis for specific BICMB systems which do not meet the design criteria by quantifying the error patterns via the adapted transfer function.

BICMB consists of parallel channels whose squared coefficients have the pdf of the eigenvalues of the uncorrelated central Wishart matrices[1] generated by MIMO fading channel. The parallel channel structure in [8], [10], [11], and [12], which provides performance analysis of the SISO coded system over a block fading channel, is the same as that of the equivalent channel model of BICMB, except that the coefficients of the parallel channels are assumed to be identically independent Rician or Rayleigh distributed. Contrary to diversity analysis for the SISO coded system in [10] and [12], PEP calculation of BICMB requires a marginal pdf of a subset of the ordered eigenvalues of the uncorrelated central Wishart matrices. Although the closed form expression of the marginal pdf is already shown in [13], it is not useful in diversity analysis for two reasons. First, it is in the form of a product of integrals to be calculated (what is stated as a tensor operator). Second, the integrals to be calculated result in a large number of recursive integral expressions. We will calculate this marginal pdf using the joint pdf available in the literature. In fact, we start with a similar set of recursive integral expressions, and simplify them to calculate our simple bound.

Based on the diversity analysis, the maximum achievable diversity order is analyzed for given code rate and the number of subchannels. For this purpose, we use a technique similar to that used in [10] and [12]. Diversity order is determined by a parameter that is inherited from the BICMB configuration. Through the Singleton bound, we show the relation between this parameter, code rate, and the number of subchannels. We note that the authors of [14] recently calculated this bound by applying the Matryoshka block fading channel model defined in [15] to a system equivalent to BICMB. Finally, the design rule of a spatial interleaver to get the maximum achievable diversity order is proposed.

II. BICMB Overview

The code rate $R_c = k_c/n_c$ convolutional encoder, possibly combined with a perforation matrix for a high rate punctured

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[1]A central Wishart matrix is the Hermitian matrix $\mathbf{AA}^H$ where the entry of the matrix $\mathbf{A}$ is complex Gaussian with zero mean so that $E[\mathbf{A}] = \mathbf{0}$. The Wishart matrix $\mathbf{AA}^H$ is called uncorrelated if the common covariance matrix, defined as $\mathbf{C} = E[\mathbf{a}_s\mathbf{a}_s^H]\,\forall s$, where $\mathbf{a}_s$ is the s^{th} column vector of $\mathbf{A}$, satisfies $\mathbf{C} = \mathbf{I}$ [9].

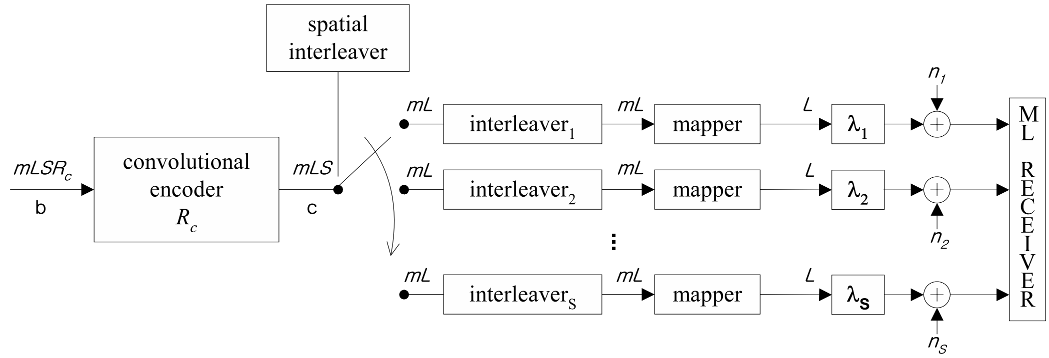

Fig. 1. Equivalent system model of BICMB.

code, generates the codeword $\mathbf{c}$ from the information vector. Then, the spatial interleaver distributes the coded bits into S sequences, each of which is interleaved by a bit-wise interleaver. The interleaved sequences are mapped by Gray encoding onto the symbol sequences. A symbol belongs to a signal set $\chi \subset \mathbb{C}$ of size $|\chi| = 2^m$, such as 2^m-QAM, where m is the number of input bits to the Gray encoder.

The MIMO channel $\mathbf{H} \in \mathbb{C}^{M\times N}$ is assumed to be quasi-static, Rayleigh, and flat fading, and perfectly known to both the transmitter and the receiver. In this channel model, we assume that the channel coefficients remain constant for the L symbol duration. The beamforming vectors are determined by the singular value decomposition of the MIMO channel, i.e., $\mathbf{H} = \mathbf{U\Lambda V^H}$ where $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices, and $\mathbf{\Lambda}$ is a diagonal matrix whose s^{th} diagonal element, $\lambda_s \in \mathbb{R}^+$, is a singular value of $\mathbf{H}$ in decreasing order. When S symbols are transmitted at the same time, then the first S vectors of $\mathbf{U}$ and $\mathbf{V}$ are chosen to be used as beamforming matrices at the receiver and the transmitter, respectively.

On each s^{th} subchannel at the k^{th} time instant, we get

$$r_{k,s} = \lambda_s y_{k,s} + n_{k,s} \quad (1)$$

where $r_{k,s}$, $y_{k,s}$ are the detected and the transmitted symbol, respectively. The term $n_{k,s}$ is additive white Gaussian noise with zero mean and variance $N_0 = N/SNR$, and $\mathbf{H}$ is complex Gaussian with zero mean and unit variance. To make the received signal-to-noise ratio SNR, the total transmitted power is scaled as N. The equivalent system model is shown in Fig. 1[2].

The location of the l^{th} coded bit c_l within the detected symbols is stored in a table $l \to (k, s, i)$, where k, s, and i are time instant, subchannel, and bit position on a label, respectively. Let $\chi_b^i \subset \chi$ where $b \in \{0, 1\}$ in the i^{th} bit position. By using the information in the table and the input-output relation in (1), the receiver calculates the ML bit metrics as

$$\gamma^i(r_{k,s}, c_l) = \min_{y\in\chi_{c_l}^i} |r_{k,s} - \lambda_s y|^2. \quad (2)$$

[2]The temporal interleavers in Fig. 1 are needed due to BICM, and the spatial interleaver due to spatial multiplexing. Conceptually, the spatial and the temporal interleavers in this figure can be combined to generate a single, more complicated interleaver. However, the explicit separation of spatial and temporal interleavers in Fig. 1 is more in line with the analysis in this paper and previous papers on the subject.

Finally, the ML decoder makes decisions according to the rule

$$\hat{\mathbf{c}} = \arg\min_{\tilde{\mathbf{c}}} \sum_l \gamma^i(r_{k,s}, \tilde{c}_l). \quad (3)$$

III. α-SPECTRA

The BER of a BICMB system is upper bounded by the summations of each PEP for all of the error events on the trellis [5], [6]. Therefore, the calculation of PEP for each error event is needed to analyze the diversity order of a given BICMB system. If the interleaver is properly designed such that the consecutive long coded bits are mapped onto distinct symbols, the PEP between the two codewords $\mathbf{c}$ and $\hat{\mathbf{c}}$ with Hamming distance d_H is upper bounded as [5]

$$\begin{aligned} P(\mathbf{c} \to \hat{\mathbf{c}}) &= E\left[P(\mathbf{c} \to \hat{\mathbf{c}}|\mathbf{H})\right] \\ &\le E\left[\frac{1}{2}\exp\left(-\frac{d_{min}^2 \sum_{s=1}^{S} \alpha_s \lambda_s^2}{4N_0}\right)\right] \end{aligned} \quad (4)$$

where d_{min} is the minimum Euclidean distance in the constellation and α_s denotes the number of times the s^{th} subchannel is used corresponding to d_H bits under consideration, satisfying $\sum_{s=1}^{S} \alpha_s = d_H$. Since PEP is affected by the summation of the products between α_s and singular values as is seen in (4), it is important to calculate the α-vector, which we define as $[\alpha_1 \cdots \alpha_S]$, for each error path to have an insight into the diversity order behavior of a particular BICMB implementation.

It has been shown in [5], [6] that for a single-carrier BICMB system, if the interleaver is designed such that, for all error paths of interest with Hamming distance d_H to the all-zeros path,

1) the consecutive coded bits are mapped over different symbols,
2) $\alpha_s \ge 1$ for $1 \le s \le S$,

then the BICMB system achieves full diversity. In this paper, we will analyze cases where the sufficient condition $\alpha_s \ge 1$ may not be satisfied, i.e., $\alpha_s = 0$ for some $s = 1, 2, \cdots, S$ is possible. In order to carry out this analysis, as well as to get an insight into the system behavior in [5], [6], one needs a method to calculate the values of α_s of an error path at Hamming distance d_H to the all-zeros path.

The α-vectors can be found by the generalized transfer function which is suggested in [7], [8], [16], and [17]. Using this method, we illustrate the α-vectors from the generalized transfer functions of a 4-state $1/2$-rate convolutional code with generator polynomials (5, 7) in octal notation, combined with several different spatial interleavers in (5), (6), and (7). The spatial interleaver used in $\mathbf{T}_1$ and $\mathbf{T}_2$ is a simple rotating switch on 2 and 3 subchannels, respectively. For $\mathbf{T}_3$, the i^{th} coded bit is de-multiplexed into subchannel $s_{\text{mod}(i-1,18)+1}$ where $s_1 = \cdots = s_6 = 1$, $s_7 = \cdots = s_{12} = 2$, $s_{13} = \cdots = s_{18} = 3$ and mod is the modulo operation. Throughout the transfer functions, the power of Z in each term of this series indicates the Hamming distance d_H associated with an error path, and the powers of a, b, c indicate how may times a particular subchannel has an errored bit in that path. The variables a, b, and c represent 1^{st}, 2^{nd}, and 3^{rd} subchannel, respectively, in a decreasing order of singular values from the channel matrix.

$$\begin{aligned}\mathbf{T}_1 &= Z^5(a^2b^3) + Z^6(a^4b^2 + a^2b^4) \\ &+ Z^7(3a^4b^3 + a^2b^5) \\ &+ Z^8(a^6b^2 + 6a^4b^4 + a^2b^6) \\ &+ Z^9(5a^6b^3 + 10a^4b^5 + a^2b^7) \\ &+ Z^{10}(a^8b^2 + 15a^6b^4 + 15a^4b^6 + a^2b^8) + \cdots\end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{T}_2 &= Z^5(a^2b^2c + a^2bc^2 + ab^2c^2) \\ &+ Z^6(a^3b^2c + a^2b^3c + a^3bc^2 + \\ &\quad ab^3c^2 + a^2bc^3 + ab^2c^3) \\ &+ Z^7(2a^3b^3c + 2a^3b^2c^2 + 2a^2b^3c^2 + \\ &\quad 2a^3bc^3 + 2a^2b^2c^3 + 2ab^3c^3) \\ &+ Z^8(a^5b^3 + a^4b^3c + a^3b^4c + 2a^4b^2c^2 + \\ &\quad 3a^3b^3c^2 + 2a^2b^4c^2 + a^4bc^3 + 3a^3b^2c^3 + \\ &\quad 3a^2b^3c^3 + ab^4c^3 + b^5c^3 + a^3bc^4 + \\ &\quad 2a^2b^2c^4 + ab^3c^4 + a^3c^5) + \cdots\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbf{T}_3 &= Z^5(a^5 + a^3b^2 + a^2b^3 + \\ &\quad b^5 + a^3c^2 + b^3c^2 + a^2c^3 + b^2c^3 + c^5) \\ &+ Z^6(a^4b^2 + 3a^3b^3 + a^2b^4 + a^4c^2 + 3a^2b^2c^2 + \\ &\quad b^4c^2 + 3a^3c^3 + 3b^3c^3 + a^2c^4 + b^2c^4) \\ &+ Z^7(2a^4b^3 + 2a^3b^4 + a^3b^3c + 7a^3b^2c^2 + \\ &\quad 7a^2b^3c^2 + 2a^4c^3 + a^3bc^3 + 7a^2b^2c^3 + \\ &\quad ab^3c^3 + 2b^4c^3 + 2a^3c^4 + 2b^3c^4) + \cdots\end{aligned} \tag{7}$$

$\mathbf{T}_1$ shows no term that lacks any of variables a and b, which means the interleaver satisfies the full diversity order criterion, $\alpha_s \geq 1$ for s = 1, 2 [5], [6]. Most of the terms in $\mathbf{T}_2$ are comprised of three variables, a, b, and c. However, three error events with Hamming distance of 8 lack one variable, resulting in the α-vectors as [5 3 0], [0 5 3], and [3 0 5]. In $\mathbf{T}_3$, many terms missing one or two variables are observed. Especially, vectors with $\alpha_s = 0$ for two subchannels are found to be [5 0 0], [0 5 0], and [0 0 5]. In Section IV, we present how these vectors affect the diversity order of BICMB.

IV. Diversity Analysis

Through the transfer functions in Section III, we have seen interleavers which do not guarantee the full diversity criteria. As stated previously, contrary to the assumption in [5] that $\alpha_s \geq 1$ for $s = 1, 2, \cdots, S$, we assume in this paper that it is possible to have $\alpha_s = 0$ for some $s = 1, 2, \cdots, S$. Let us define α_{nzmin} as the minimum α among the nonzero α's in the α-vector. Using the inequality $\sum_{s=1}^{S} \alpha_s \lambda_s^2 \geq \alpha_{nzmin} \sum_{k=1,\alpha_k \neq 0}^{S} \lambda_k^2$, PEP in (4) is expressed as

$$P(\mathbf{c} \to \hat{\mathbf{c}}) \leq E\left[\frac{1}{2}\exp\left(-W\sum_{k=1}^{K}\mu_{p_k}\right)\right] \tag{8}$$

where $W = d_{min}^2 \alpha_{nzmin}/(4N_0)$, and μ_s is the squared singular value of the s^{th} subhcannel (λ_s^2). In (8), p_k is the k^{th} element of a vector $\mathbf{p} = [p_1 \cdots p_K]^T$ whose elements are the indices corresponding to non-zero α, i.e., $\alpha_{p_k} \neq 0$. Similarly, $\mathbf{s} = [s_1 \cdots s_{Y-K}]^T$ is defined as a vector whose elements are the indices k such that $\alpha_{s_k} = 0$. Throughout this paper, we use X and Y as $X = \max(N, M)$, and $Y = \min(N, M)$. The vectors $\mathbf{p}$ and $\mathbf{s}$ are sorted in an increasing order. To calculate (8), we need the marginal pdf of the K eigenvalues by calculating the multiple integration over the domain $\mathcal{D}_{\mathbf{s}}$

$$f(\mu_{p_1}, \cdots, \mu_{p_K}) = \int \cdots \int_{\mathcal{D}_{\mathbf{s}}} \rho(\mu_1, \cdots, \mu_Y)\, d\mu_{s_{Y-K}} \cdots d\mu_{s_1}. \tag{9}$$

The joint pdf of the ordered strictly positive eigenvalues of the uncorrelated central Wishart matrices $\rho(\mu_1, \cdots, \mu_Y)$ in (9) is available in the literature [18], [19] as

$$\rho(\mu_1, \cdots, \mu_Y) = \prod_{i=1}^{Y} \mu_i^{X-Y} \prod_{j>i}^{Y} (\mu_i - \mu_j)^2 e^{-\sum_{j=1}^{Y} \mu_j}. \tag{10}$$

Because we are interested in the exponent of W, the constant, which appears in the literature, is ignored in (10) for brevity.

The evaluation of the marginal pdf is complicated due to the multiple integration of the product of the polynomial and the exponential function in (9), (10). The complexity mainly comes from the fact that the elementary integration inside the multiple integration, which is $\int_0^x y^m e^{-y} dy$, generates a large number of terms of the form $x^n e^{-x}$ for large m. However, if we remove the exponential function of the elementary integration, the integration produces only one term, resulting in a much simpler multiple integration. In addition, since the eigenvalues of the Wishart matrix are positive and real, $e^{-\mu_i} \leq 1$ holds true for any i. This idea leads to a simple result of the elementary integration as $\int_0^x y^m e^{-y} dy \leq \frac{1}{m+1} x^{m+1}$.

To apply the idea above to the calculation of the marginal pdf, we introduce an upper bound to the joint pdf as

$$\hat{\rho}(\mu_1, \cdots, \mu_Y) = \begin{cases} \psi(\mu_1, \cdots, \mu_Y)\, e^{-\left(\mu_1 + \sum_{k=1}^{K} \mu_{p_k}\right)} & \text{if } \alpha_1 = 0 \\ \psi(\mu_1, \cdots, \mu_Y)\, e^{-\sum_{k=1}^{K} \mu_{p_k}} & \text{if } \alpha_1 > 0 \end{cases} \tag{11}$$

where the polynomial $\psi(\mu_1, \cdots, \mu_Y)$ is defined as $\psi(\mu_1, \cdots, \mu_Y) = \prod_{i=1}^{Y} \mu_i^{X-Y} \prod_{j>i}^{Y} (\mu_i - \mu_j)^2$. By replacing $\rho(\mu_1, \cdots, \mu_Y)$ with $\hat{\rho}(\mu_1, \cdots, \mu_Y)$ in (9), we get the upper-bound to the marginal pdf in Theorem 1, emphasizing the smallest degree of the terms because it plays an important role in determining the behavior of (8) in the high signal-to-noise ratio regime.

Theorem 1: The marginal pdf $f(\mu_{p_1}, \cdots, \mu_{p_K})$ is upper bounded as

$$f(\mu_{p_1}, \cdots, \mu_{p_K}) \le r(\mu_{p_1}, \cdots, \mu_{p_K}) e^{-\sum_{k=1}^{K} \mu_{p_k}}$$

where $r(\mu_{p_1}, \cdots, \mu_{p_K})$ is a multivariate polynomial whose smallest degree is $(N-Q+1)(M-Q+1)-K$, and Q is defined as an index indicating the first non-zero α, i.e., $Q = p_1$.

Proof: See [17]. □

In the case of the single eigenvalue where $K = 1$, and $p_1 = l$, Theorem 1 states that the smallest degree of μ_l is $(N-l+1)(M-l+1)-1$. This generalizes the result of the first order expansion in [4], [20] to calculate the marginal pdf of the l^{th} eigenvalue.

We are now ready to calculate the expectation of (8) by calculating

$$\begin{aligned} E\left[\exp\left(-W\sum_{k=1}^{K}\mu_{p_k}\right)\right] &= \int \cdots \int_{\mathcal{D}_{\mathbf{p}}} r(\mu_{p_1}, \cdots, \mu_{p_K}) \\ &\times e^{-(1+W)\sum_{k=1}^{K}\mu_{p_k}} d\mu_{p_K} \cdots d\mu_{p_1} \end{aligned} \quad (12)$$

where $\mathcal{D}_{\mathbf{p}}$ is the domain of integration. Note that $1+W \approx W$ for high signal-to-noise ratio. In addition, it can be easily verified that the following equality of a specific term in the polynomial for a domain $\infty > \nu_1 > \nu_2 > \cdots > \nu_K > 0$ holds true;

$$\int_0^{\infty} \cdots \int_0^{\nu_{K-1}} \nu_1^{\beta_1} \cdots \nu_K^{\beta_K} e^{-\omega \sum_{k=1}^{K} \nu_k} d\nu_K \cdots d\nu_1 = \zeta \omega^{-\left(K+\sum_{k=1}^{K}\beta_k\right)} \quad (13)$$

where ζ is a constant. Since the polynomial $r(\mu_{p_1}, \cdots, \mu_{p_K})$ is the sum of a number of terms with different degrees, the result of (12) is also the sum of the terms of W whose exponent is the corresponding degree. For large W, it is easy to see that the overall sum is dominated by the terms with the smallest degree of W^{-1}, which results from the smallest degree of $r(\mu_{p_1}, \cdots, \mu_{p_K})$. Therefore, we conclude that (8) is upper bounded by

$$E\left[\exp\left(-W\sum_{k=1}^{K}\mu_{p_k}\right)\right] \le \eta W^{-(N-Q+1)(M-Q+1)} \quad (14)$$

where η is a constant. Since BER is dominated by PEP with the worst exponent term, the diversity order of a given BICMB system is $(N-Q_{max}+1)(M-Q_{max}+1)$, where Q_{max} is defined as the maximum among Q's corresponding to all of the error events.

V. Diversity Order Bound

In this section, we show the diversity order bound by employing the Singleton bound. Let us define $d_{E,s}(\mathbf{c}, \hat{\mathbf{c}})$ as the Euclidean distance between the mapped symbols of the two codewords residing on the s^{th} subchannel, $d_{E,s}(\mathbf{c}, \hat{\mathbf{c}}) = \sum_{k=1}^{L} |y_{k,s} - \hat{y}_{k,s}|^2$ where $y_{k,s}$ and $\hat{y}_{k,s}$ are the symbols on the s^{th} subchannel at the time index k from the codewords $\mathbf{c}$ and $\hat{\mathbf{c}}$, respectively. If α_s is equal to zero, then all of the coded bits on the s^{th} subchannel of the two codewords are the same. Since we assume that the consecutive bits are mapped over different symbols, the symbols corresponding to the same coded bits of the s^{th} subchannel are also the same, resulting in $d_{E,s}(\mathbf{c}, \hat{\mathbf{c}}) = 0$. Then, the parameter Q can be viewed as an index to the first non-zero element in a vector $[d_{E,1}(\mathbf{c}, \hat{\mathbf{c}}) \quad d_{E,2}(\mathbf{c}, \hat{\mathbf{c}}) \quad \cdots \quad d_{E,S}(\mathbf{c}, \hat{\mathbf{c}})]$. In the case of a pair of the codewords that has $S-1$ non-zero $d_{E,s}(\mathbf{c}, \hat{\mathbf{c}})$'s, Q can be 2 because of the vector type $[0 \times \times \cdots \times]$, or 1 from $[\times\, 0 \times \times \cdots \times]$, $[\times \times 0 \times \times \cdots \times]$, $\cdots$, $[\times \times \times \cdots \times 0]$, where $\times$ stands for non-zero value. In general, for a pair of the codewords that has δ_H non-zero $d_{E,s}(\mathbf{c}, \hat{\mathbf{c}})$'s, Q is bounded as

$$Q \le S - \delta_H + 1. \quad (15)$$

If we consider the L symbols transmitted on each subchannel as a super-symbol over χ^L, then the transmitted symbols for all the subchannels in a block can be viewed as a vector of length S super-symbols. For convenience, we call this vector of super-symbols as a symbol-wise codeword. We now introduce a distance between $\mathbf{c}$ and $\hat{\mathbf{c}}$, which we call δ_H, as the number of non-zero elements in the vector $[d_{E,1}(\mathbf{c}, \hat{\mathbf{c}}) \quad d_{E,2}(\mathbf{c}, \hat{\mathbf{c}}) \quad \cdots \quad d_{E,S}(\mathbf{c}, \hat{\mathbf{c}})]$. This distance is similar to the Hamming distance but it is between two non-binary symbol-wise codewords. By using the Singleton bound which is also applicable to non-binary codes, we can calculate the minimum distance of the symbol-wise codewords in a way similar to finding the minimum Hamming distance of binary codes. Let us define $\mathcal{M}$ as the number of distinct symbol-wise codewords. Then we see that $\mathcal{M} = 2^{mLSR_c}$ from Fig. 1. Let k $(0 < k \le S-1)$ denote the integer value satisfying $2^{mL(k-1)} < \mathcal{M} \le 2^{mLk}$. Since $\mathcal{M} > 2^{mL(k-1)}$, there necessarily exist two symbol-wise codewords whose $k-1$ elements are the same. From the Singleton bound [21], the minimum distance of these symbol-wise codewords $\delta_{H,min}$ is expressed as $\delta_{H,min} \le S - k + 1$. Since $2^{mLSR_c} \le 2^{mL(S-\delta_{H,min}+1)}$, we get

$$\delta_{H,min} \le S - \lceil S \cdot R_c \rceil + 1 \quad (16)$$

using the fact that $\delta_{H,min}$ is an integer value.

For a given BICMB system with $\delta_{H,min}$, it is true that the distance δ_H between any pair of the codewords is always larger than or equal to the minimum distance $\delta_{H,min}$. By combining the inequalities of $\delta_H \ge \delta_{H,min}$ and (15), we get $\delta_{H,min} \le \delta_H \le S - Q + 1$, leading to $Q \le S - \delta_{H,min} + 1$. From this inequality, the maximum Q is decided as

$$Q_{max} = S - \delta_{H,min} + 1. \quad (17)$$

The inequality (16) and the equation (17) result in the inequality as $Q_{max} \ge \lceil S \cdot R_c \rceil$. Taking the result of the diversity analysis in Section IV into account, we get the maximum achievable diversity order as $(N - \lceil S \cdot R_c \rceil + 1)(M - \lceil S \cdot R_c \rceil + 1)$.

Since we assumed in the previous description that there exist a convolutional encoder and a spatial interleaver which satisfy the relation $Q_{max} = \lceil S \cdot R_c \rceil$, we will show the specific design method of the interleaver from a given convolutional encoder to ensure the relation. The following method is not the unique solution to guarantee the maximum achievable condition, but simple to state the concept. Consider a BICMB system with S subchannels and the code rate $R_c = k_c/n_c$ convolutional code. Each of $P = LCM(n_c, S)$ coded bits is distributed to the S streams in the order specified by the interleaving pattern, where $LCM(\cdot)$ is the least common multiple operation. Since each stream needs to be evenly employed for a period, P/S coded bits are assigned on each stream. To guarantee $Q_{max} = \lceil S \cdot R_c \rceil$, it is sufficient to consider only the first branches that split from the zero state in one period because of the repetition property of the convolutional code. We incorporate the basic idea that once the s^{th} stream is assigned to an error bit of the first branch, all of the error events containing that branch give $\alpha_s > 0$, resulting in $Q \leq s$. By extending this idea, we summarize the assignment procedure as

1) the lowest available subchannel is assigned to the error bit position of one of the first branches which have not yet assigned to any subchannel,
2) the procedure 1) is repeated until all of the first branches are assigned to one of the subchannels. If all of the first branches are assigned to one of the subchannels, the assignment procedure quits after the rest of subchannels are assigned randomly to the unassigned bit positions, subject to satisfying the rate condition on each subchannel.

An example of the design method is provided in [17].

VI. Simulation Results

To show the validity of the diversity order analysis in Section IV using the parameter Q_{max}, BER against SNR are derived through a Monte-Carlo simulation[3]. Fig. 2 shows BER performances for the cases corresponding to $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ in (5), (6), and (7). The well-known reference curves achieving the full diversity order of NM are drawn from the Alamouti code for the 2×2 case and $1/2$-rate orthogonal space-time block code (OSTBC) for the 3×3 case [22]. From (5), Q_{max} for $\mathbf{T}_1$ is found to be 1 because $\alpha_s \geq 1$ for $s = 1, 2$ in all of the α-vectors. In this case, as predicted by the analysis in [5], [6], the diversity order equals 4. From the figure, we see that BER curve for $\mathbf{T}_1$ is parallel to that of 2×2 Alamouti code. Since Q_{max} for $\mathbf{T}_2$ is 2 due to the vector $[0\,5\,3]$, the calculated diversity order is 4 in the case of $M = N = S = 3$. This is verified by Fig. 2, losing the full diversity order 9. Although the same number of subchannels and the same convolutional code as for $\mathbf{T_2}$ are used, the different spatial interleaver from that of $\mathbf{T_2}$, described in Section III for $\mathbf{T_3}$, gives no diversity gain at all. The reason for this is that the vector $[0\,0\,5]$ which is observed from the transfer function in $\mathbf{T}_3$ makes Q_{max} 3 resulting in the calculated diversity order of 1. This matches the simulation result.

[3]For a discussion on the use of BER to determine the diversity order of BICMB, please refer to the text around equations (16)-(18) in [5].

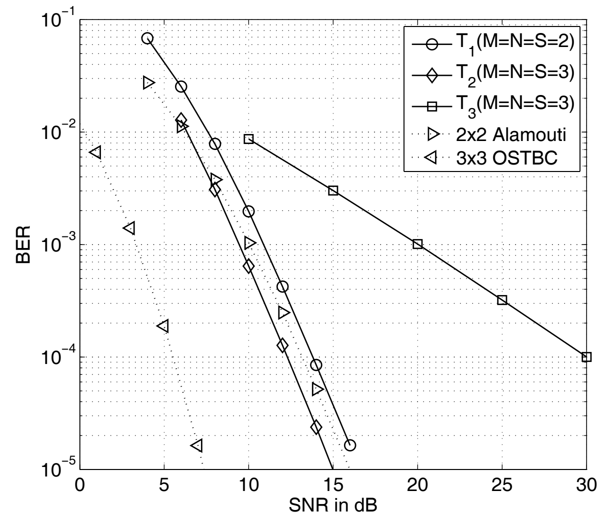

Fig. 2. Simulation results for a 4-state 1/2-rate convolutional code with different spatial interleavers. 4-QAM is used for all of the curves.

TABLE I
Search Results of the Dominant α-Vectors for 64-State Convolutional Codes

S	rate	d_{free}	dominant α-vectors	Q_{max}
2	$1/2$	10	[3 7] [4 6] [5 5]	1
	$2/3$	6	[0 12] [0 14] [0 15]	2
	$3/4$	5	[0 8] [0 10] [0 12]	2
3	$1/3$	15	[3 6 6] [5 4 6] [4 6 6]	1
	$1/2$	10	[0 7 7] [0 8 6] [0 9 7]	2
	$2/3$	6	[0 4 5] [0 6 3] [0 4 6]	2
	$3/4$	5	[0 0 13] [0 0 15] [0 0 17]	3

Table I shows results of a computer search of the α-vectors of BICMB with industry standard 64-state convolutional codes and a simple rotating spatial interleaver. The generator polynomials for rates $1/2$ and $1/3$ are (133, 171) and (133, 145, 175) in octal, respectively. For the high rate codes such as $2/3$ and $3/4$, the perforation matrices in [23] are used from the $1/2$-rate original code. Instead of displaying the whole transfer functions, we present only three α-vectors among such a number of dominant α-vectors that lead to Q_{max}. The search results comply with the bound $Q_{max} \geq \lceil S \cdot R_c \rceil$ as was analyzed in Section V.

Fig. 3 shows the BER performance of the 2×2 $S = 2$ BICMB system with the 64-state convolutional code and a simple rotating spatial interleaver. The diversity orders of the systems with punctured codes are 1 because both Q_{max} values corresponding to the codes shown in Table I are 2, while the system with the $1/2$-rate convolutional code, whose Q_{max} is equal to 1, achieves the full diversity order of 4.

As shown in Fig. 4, for a 3×3 system with 3 streams, only $1/3$-rate convolutional code achieves the full diversity order of 9 since other codes have Q_{max} of larger than 1 as given in Table I. The analytically calculated diversity orders by using Q_{max} in Table I are 4, 4, 1 for $1/2$, $2/3$, $3/4$ respectively, which are easily verified from Fig. 4.

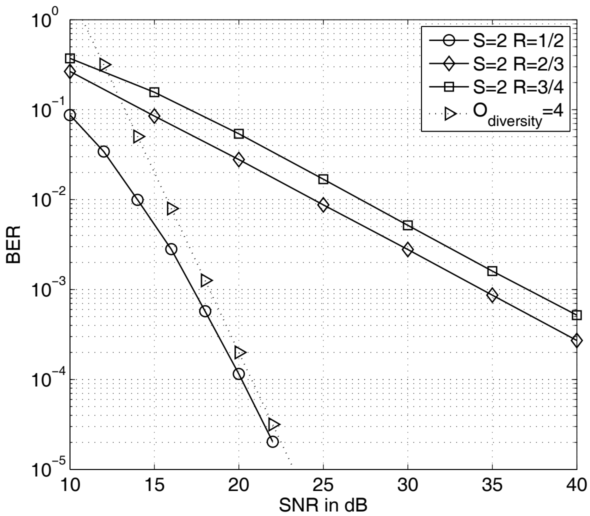

Fig. 3. Simulation results for the 2×2 case where 16-QAM is used for all of the curves.

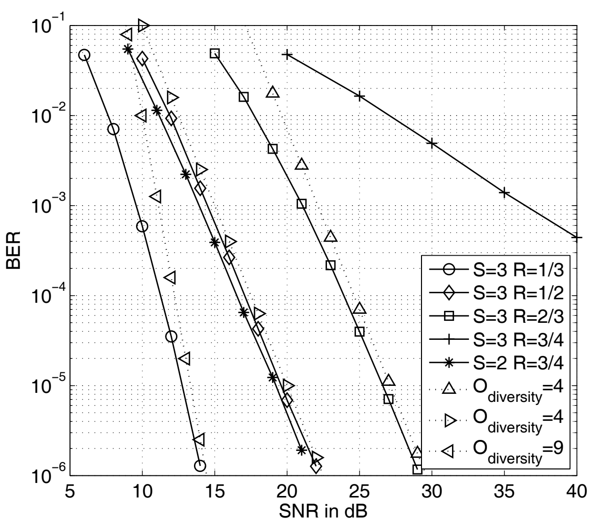

Fig. 4. Simulation results for 3×3 case where 16-QAM is used for all of the curves.

VII. Conclusion

In this paper, we investigated the diversity order of BICMB when the interleaver does not meet the previously introduced design criteria. By using the generalized transfer function method, the α-vectors that do not meet the full diversity order criteria are quantified. Then, the diversity behavior corresponding to the α-vectors was analyzed through PEP calculation. We presented a method to get an upper bound to the marginal pdf of the eigenvalues by simplifying the multiple integration. As a result, the exponent of PEP between two codewords is $(N - Q + 1)(M - Q + 1)$ where Q is an index indicating the first non-zero element in the α-vector. Since BER is dominated by PEP with the smallest exponent, the diversity order is $(N - Q_{max} + 1)(M - Q_{max} + 1)$, where Q_{max} is the maximum among Q's corresponding to all of the α-vectors. We provided the simulation results that verify the analysis. We also showed that Q_{max} is lower bounded by the product of the code rate and the number of streams, leading to the maximum achievable diversity order equal to $(N - \lceil S \cdot R_c \rceil + 1)(M - \lceil S \cdot R_c \rceil + 1)$. Finally, we proposed the design rule of the spatial interleaver to guarantee the maximum achievable diversity order.

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