A Novel Maximum Likelihood Decoding Algorithm for Orthogonal Space-Time Block Codes

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***Abstract*—In this letter, we propose a low complexity Maximum Likelihood (ML) decoding algorithm for orthogonal space-time block codes (OSTBCs) based on the real-valued lattice representation and QR decomposition. We show that for a system with rate $r = K/T$, where K is the number of transmitted symbols per T time slots, the proposed algorithm decomposes the original complex-valued system into a parallel system represented by $2K$ real-valued components, thus allowing for a simple and independent detection of the real and imaginary parts of each complex transmitted symbol. We further show that for square L-QAM constellations, the proposed algorithm reduces the decoding computational complexity from $\mathcal{O}(L)$ for conventional ML to $\mathcal{O}(\sqrt{L})$ without sacrificing the performance.**

***Index Terms*—Maximum likelihood (ML) decoding, orthogonal space-time block codes (OSTBCs), QR decomposition.**

I. Introduction

SPACE-TIME block codes (STBCs) from orthogonal designs (OSTBCs) are attractive since they achieve the maximum diversity, the maximum coding gain, and the highest throughput [1]. These codes are used in multiple-input multiple-output (MIMO) systems to introduce high performance gains [2]. Their design allows simple Maximum Likelihood (ML) decoding. The decoding complexity is very critical for practical employment of MIMO systems. In OSTBCs proposed by Alamouti [3] and Tarokh *et al.* [4], each transmitted symbol is decoded separately, resulting in linear decoding complexity.

For N transmit antennas, a complex orthogonal space-time block code is described by a $T \times N$ transmission matrix $\mathcal{G}_N$, where each entry in $\mathcal{G}_N$ is a linear combination of the K variables $s_1, s_2, \ldots, s_K$ and their conjugates [5]. $\mathcal{G}_N$ can send K symbols from a signal constellation in a block of T channel uses. Since T time slots are used to transmit K symbols, the rate of $\mathcal{G}_N$ is defined as $r = K/T$ [6].

In this letter, we focus on the decoding complexity of OSTBCs. We introduce a new decoding algorithm for square QAM constellations based on the QR decomposition of the real-valued lattice representation and show that conventional simple ML detection can be further simplified. In other words, we show that the optimal ML performance for OSTBCs is obtained with a substantial reduction in the decoding complexity. We also compare our decoding complexity with that of conventional ML detection.

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The remainder of this letter is organized as follows: In Section II, we specify the system model and define the problem. In Section III, we introduce the new decoding algorithm. A complexity discussion is provided in Section IV. Finally, we conclude the letter in Section V.

II. System Model and Problem Definition

Consider a MIMO system with N transmit and M receive antennas, and an interval of T symbols during which the channel is constant. The received signal is given by

$$Y = \mathcal{G}_N H + V \tag{1}$$

where $Y = [y_t^j]_{T \times M}$ is the received signal matrix of size $T \times M$ and whose entry y_t^j is the signal received at antenna j at time t, $t = 1, 2, \ldots, T$, $j = 1, 2, \ldots, M$; $V = [v_t^j]_{T \times M}$ is the noise matrix, and $\mathcal{G}_N = [g_t^i]_{T \times N}$ is the transmitted signal matrix whose entry g_t^i is the signal transmitted at antenna i at time t, $t = 1, 2, \ldots, T$, $i = 1, 2, \ldots, N$. The matrix $H = [h_{i,j}]_{N \times M}$ is the channel coefficient matrix of size $N \times M$ whose entry $h_{i,j}$ is the channel coefficient from transmit antenna i to receive antenna j. The entries of the matrices H and V are independent, zero-mean, and circularly symmetric complex Gaussian random variables of unit variance.

Assuming that the channel H is known at the receiver, the ML estimate is obtained at the decoder by performing $\min_{\mathcal{G}_N} ||Y - \mathcal{G}_N H||_F^2$, where $||.||_F$ is the Frobenius norm. OSTBCs have a very simple and decoupled ML decoding algorithm. The squared norm $||Y - \mathcal{G}_N H||_F^2$ can be decoupled into K parts, where each part decodes one transmitted symbol independently [5]. We illustrate this by an example. Consider the OSTBC proposed by Alamouti [3] for $N = 2$ and defined as

$$\mathcal{G}_2 = \begin{pmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{pmatrix}. \tag{2}$$

The receiver decodes s_1 and s_2 by decomposing the measure $||Y - \mathcal{G}_N H||_F^2$ into two parts, and minimizes each separately over all possible values of s_1 and s_2 that belong to the constellation used. Let the square L-QAM alphabet be given as Ω^2, where $\Omega = \{-\sqrt{L}+1, -\sqrt{L}+3, \ldots, \sqrt{L}-1\}$. Then, ML is equivalent to [4]

$$\hat{s}_1 = \arg\min_{s_1 \in \Omega^2} \left| \left[\sum_{j=1}^{M} \left(y_1^j h_{1,j}^* + (y_2^j)^* h_{2,j} \right) \right] - s_1 \right|^2 + \left(-1 + \sum_{j=1}^{M} \sum_{i=1}^{2} |h_{i,j}|^2 \right) |s_1|^2$$

and,

$$\hat{s}_2 = \arg\min_{s_2\in\Omega^2}\left|\left[\sum_{j=1}^{M}\left(y_1^j h_{2,j}^* - (y_2^j)^* h_{1,j}\right)\right] - s_2\right|^2 + \left(-1+\sum_{j=1}^{M}\sum_{i=1}^{2}|h_{i,j}|^2\right)|s_2|^2.$$

Obviously, ML detection in [5] is simple since it decodes each transmitted symbol independently. A number of ML decoders for $N > 2$ were derived in [5]. In a similar way, it was shown that the decoder decomposes the ML measure into K parts where each is minimized over all constellation points to decode one symbol separately. As a result, the complexity of ML decoding in [5] is $\mathcal{O}(L)$ which is linear with the constellation size L. Thus, the decoding algorithm can be implemented using only linear processing at the receiver. We will show in this letter that the complexity can still be reduced substantially. The algorithm proposed in this letter reduces the decoding complexity from $\mathcal{O}(L)$ to $\mathcal{O}(\sqrt{L})$ with a substantial reduction in the number of arithmetic operations required.

III. Proposed Algorithm

We start by rewriting (1) in matrix form

$$\begin{bmatrix} y_1^1 & \cdots & y_1^M \\ \vdots & \ddots & \vdots \\ y_T^1 & \cdots & y_T^M \end{bmatrix} = \mathcal{G}_N H + \begin{bmatrix} v_1^1 & \cdots & v_1^M \\ \vdots & \ddots & \vdots \\ v_T^1 & \cdots & v_T^M \end{bmatrix} \quad (3)$$

where

$$H = \begin{bmatrix} h_{1,1} & \cdots & h_{1,M} \\ \vdots & \ddots & \vdots \\ h_{N,1} & \cdots & h_{N,M} \end{bmatrix}.$$

We specify the complex transmitted symbols $s_1, s_2, \ldots, s_K$ of $\mathcal{G}_N$ by their real and imaginary parts as $s_i = x_{2i-1} + jx_{2i}$ for $i = 1, 2, \ldots, K$. Now, we obtain the real-valued representation of (3). To do so, we first arrange the matrices Y, H, and V, each in one column vector by stacking their columns one after the other [7] as

$$\begin{bmatrix} y_1^1 \\ \vdots \\ y_T^M \end{bmatrix} = \check{\mathcal{G}}_N \begin{bmatrix} h_{1,1} \\ \vdots \\ h_{N,M} \end{bmatrix} + \begin{bmatrix} v_1^1 \\ \vdots \\ v_T^M \end{bmatrix} \quad (4)$$

where $\check{\mathcal{G}}_N \triangleq I_M \otimes \mathcal{G}_N$, with I_M is the identity matrix of size M and $\otimes$ denoting the Kronecker matrix multiplication [7], and then we decompose the MT-dimensional complex problem defined by (4) to a $2MT$-dimensional real-valued problem by applying the real-valued lattice representation defined in [8] to obtain

$$\check{y} = \check{H}x + \check{v}$$

or equivalently

$$\begin{bmatrix} \Re(y_1^1) \\ \Im(y_1^1) \\ \vdots \\ \Im(y_T^M) \end{bmatrix} = \check{H} \begin{bmatrix} \Re(s_1) \\ \Im(s_1) \\ \vdots \\ \Im(s_K) \end{bmatrix} + \begin{bmatrix} \Re(v_1^1) \\ \Im(v_1^1) \\ \vdots \\ \Im(v_T^M) \end{bmatrix}. \quad (5)$$

The real-valued fading coefficients of $\check{H}$ are defined using the complex fading coefficients $h_{i,j}$ from transmit antenna i to receive antenna j as $h_{2l-1}^j = \Re(h_{l,j})$, and $h_{2l}^j = \Im(h_{l,j})$ for $l = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, M$. Now, since $\mathcal{G}_N$ is an orthogonal matrix and due to the real-valued representation of the system using (5), we observe that

- All columns of $\check{H} = \begin{bmatrix} \check{h}_1 & \check{h}_2 & \ldots & \check{h}_{2K} \end{bmatrix}$ where $\check{h}_i$ is the ith column of $\check{H}$, are orthogonal to each other, or equivalently
$$\langle \check{h}_i, \check{h}_j \rangle = 0, \qquad i \neq j. \quad (6)$$
- The norm of every column in $\check{H}$ is equal to the norm of any other column in $\check{H}$, i.e.,
$$\mathrm{norm}(\check{h}_i) = \mathrm{norm}(\check{h}_j), \qquad i, j = 1, 2, \ldots, 2K. \quad (7)$$

These two properties have a major impact on the complexity reduction of our proposed algorithm.

Applying QR decomposition to (5), we have

$$\begin{aligned} \check{y} &= QRx + \check{v} \\ Q^H\check{y} &= Rx + Q^H\check{v} \\ \bar{y} &= Rx + \bar{v} \end{aligned} \quad (8)$$

where $\bar{v}$ and $\check{v}$ have the same statistical properties since Q is unitary and so is Q^H. Recall that $\check{H}$ is a $2MT \times 2K$ matrix. Then Q^H is a $2K \times 2MT$ matrix and $\bar{y}$ is a one column vector of size $2K$. Since $\check{H}$ is an orthogonal matrix, QR decomposition produces a $2K \times 2K$ diagonal R matrix (see [9] for proof), a property which substantially reduces the decoding complexity.

Using (8), the ML problem is now simpler and rather than minimizing $||Y - \mathcal{G}_N H||_F^2$, the solution is obtained by minimizing the metric $||\bar{y} - Rx||_2^2$ over all different combinations of the vector x. In other words, the ML solution is found by minimizing

$$\left\|\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_{2K} \end{bmatrix} - \begin{bmatrix} r_{1,1} & 0 & \cdots & 0 \\ 0 & r_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{2K,2K} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2K} \end{bmatrix}\right\|_2^2 \quad (9)$$

over all combinations of $x \in \Omega^{2K}$. This can be further simplified as

$$\hat{x}_i = \arg\min_{x_i\in\Omega} |\bar{y}_i - r_{i,i}x_i|^2 \quad (10)$$

for $i = 1, 2, \ldots, 2K$. Then, the decoded message is

$$\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{2K})^T.$$

This means that the proposed algorithm produces $2K$ parallel 1×1 real-valued subsystems for any OSTBC, thus making the detection of the real and imaginary parts of each transmitted complex symbol possible to be carried out independently. Note that this simplification is obtained through the observation of the orthogonality properties of $\check{H}$, the observations in (6) and (7), and the QR decomposition in (8), resulting in (9) and (10). Obviously, this approach results in a simplified ML problem that can be solved in a parallel fashion to obtain the optimal solution while substantially reducing the

TABLE I
OF REAL MULTIPLICATIONS AND REAL ADDITIONS VS L FOR 2×1 SYSTEM USING ALAMOUTI CODE

	L	4	16	64	256
R_M	ML	224	896	3584	14336
	PR	128	160	224	352
R_A	ML	176	704	2816	11264
	PR	31	47	79	143

TABLE II
OF REAL MULTIPLICATIONS AND REAL ADDITIONS VS L FOR 4×1 SYSTEM USING $\mathcal{G}_4$

	L	4	16	64	256
R_M	ML	960	3840	15360	61440
	PR	732	796	924	1180
R_A	ML	864	3456	13824	55296
	PR	167	199	263	391

TABLE III
OF REAL MULTIPLICATIONS AND REAL ADDITIONS VS L FOR 3×2 SYSTEM USING $\mathcal{G}_3$

	L	4	16	64	256
R_M	ML	1606	6424	25696	102784
	PR	1388	1452	1580	1836
R_A	ML	1248	4992	19968	79872
	PR	311	343	407	535

overall decoding complexity.

IV. Computational Complexity

In this section, we compare the computational complexity of our proposed algorithm with that of conventional ML detection. The overall complexity is measured in terms of the number of operations required to decode the transmitted signals for each block period T. A complex multiplication is equivalent to 4 real multiplications R_M and 2 real additions R_A, while a complex addition is equivalent to 2 real additions. We split the complexity formula into two parts in order to represent R_M and R_A independently. We denote the complexity of our proposed algorithm by $\mathcal{C}_{PR}$, and show it as a two dimensional vector where the first dimension is the number of real multiplications and the second, the number of real additions, then

$$\mathcal{C}_{PR} = 2K\sqrt{L}(4R_M, 2R_A). \tag{11}$$

Note that performing QR decomposition requires additional number of computations. Due to the special structure of the channel matrix $\check{H}$, QR can be simplified into two simple steps. To illustrate this, let $\check{H} = \begin{bmatrix} \check{h}_1 & \check{h}_2 & \dots & \check{h}_{2K} \end{bmatrix}$ where $\check{h}_i$ is the ith column of $\check{H}$. Then, due to (6), R is diagonal. The definition of the diagonal elements in R in QR decomposition is $r_{i,i} = \mathrm{norm}(\check{h}_i)$. Due to (7) the matrices Q and R are computed by

Step 1: Calculate the diagonal elements of the matrix R by finding $r_{1,1} = \mathrm{norm}(\check{h}_1)$ and then set $r_{i,i} = r_{1,1}$ for $i = 2, \dots, 2K$. (Note that due to (7) all diagonal elements are equal).

Step 2: Calculate the unitary matrix $Q = \begin{bmatrix} q_1 & q_2 & \dots & q_{2k} \end{bmatrix}$ where $q_i = \check{h}_i / r_{i,i}$ for $i = 1, 2, \dots, 2K$.

Finding R requires $2MT + 12$ R_M and $2MT - 1$ R_A, and computing Q requires $16MTK$ R_M, assuming that a square root operation and a real division are equivalent to 12 and 4 real multiplications respectively [10].

Moreover, the computation of $\bar{y} = Q^H \tilde{y}$ requires $4MTK$ R_M and $4MTK - 2K$ R_A. Therefore, (11) is rewritten taking into account the complexity of computing QR and $Q^H \tilde{y}$ as

$$\mathcal{C}_{PR} = (MT(20K+2) + 12 + 8K\sqrt{L})R_M, \quad (MT(4K+2) + 2K(2\sqrt{L}-1) - 1)R_A. \tag{12}$$

Conventional ML detection [5], on the other hand, performs simple detection for each complex symbol independently. The complexity $\mathcal{C}_{ML}$ can be derived using the presentation in [5] (see Appendix of [5] for details) as

$$\mathcal{C}_{ML} = L((4MN(T+K) + 12K)R_M, \quad (4MN(T+K) + 6K)R_A). \tag{13}$$

Obviously, the complexity of ML is $\mathcal{O}(L)$ whereas the complexity of the proposed algorithm is $\mathcal{O}(\sqrt{L})$. Furthermore, the number of computations required to decode one block of transmitted symbols using conventional ML is much higher than that required for the proposed algorithm.

We give a comparison between $\mathcal{C}_{PR}$ and $\mathcal{C}_{ML}$ in terms of the number of real multiplication and real additions considering $N = 2, 3, 4$ for different constellation sizes. In Table I, we show this comparison for $N = 2$ considering the Alamouti OSTBC defined in (2). In Tables II and III, we show the same comparison for $N = 4$, $M = 1$ and $N = 3$, $M = 2$, respectively, using the OSTBCs $\mathcal{G}_4$ and $\mathcal{G}_3$ defined in [5].

Clearly, the complexity gain obtained by the proposed algorithm is substantial. Finally, it is important to emphasize the fact that the complexity reduction, as shown in all tables, becomes greater as L is larger.

V. Conclusions

An efficient ML decoding algorithm based on QR decomposition of the channel matrix is proposed for orthogonal space-time block codes. The performance is shown to be optimal while reducing the decoding complexity significantly compared to conventional ML. Furthermore, we show that the complexity of this algorithm is $\mathcal{O}(\sqrt{L})$ compared to $\mathcal{O}(L)$ for conventional ML, and consequently the complexity gain becomes grater as the constellation size is larger.

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