Diversity Analysis of Single and Multiple Beamforming

Ersin Sengul, *Student Member, IEEE*, Enis Akay, *Student Member, IEEE*, and Ender Ayanoglu, *Fellow, IEEE*

***Abstract*—In this letter, we study two techniques, known as single and multiple beamforming, to exploit the perfect channel state information (CSI) available both at the transmitter and the receiver of a multiantenna wireless system. Assuming N and M are the number of antennas at the transmitter and the receiver, respectively, we show that single beamforming (transmission of a single symbol from all transmit antennas at the same time, employing the best subchannel) can achieve the maximum spatial diversity order in the channel (NM). We extend our analytical results to multiple beamforming (transmission of S symbols simultaneously, $S > 1$) and calculate that the diversity order achievable for this system is $(N - S + 1)(M - S + 1)$.**

***Index Terms*—Beamforming, diversity, multi-input multi-output (MIMO) systems, pairwise error probability (PEP).**

I. Introduction

In recent years, spatial diversity techniques are under investigation to increase the robustness as well as the throughput of multi-input multi-output (MIMO) wireless systems, employing N transmit and M receive antennas [1]. These systems can be grouped into two. The first group requires the channel state information (CSI) at the receiver, but not at the transmitter. Space–time (ST) codes are a subset of these systems [2]. The second group requires perfect or partial CSI at both the transmitter and the receiver. When perfect CSI is available at both ends, two techniques that can be used are single and multiple beamforming [3]. These techniques use singular value decomposition (SVD), which separates the MIMO channel into parallel subchannels. When only the subchannel with the largest gain is used for transmission, the technique is called *single beamforming* [3]. MIMO systems can also be used to enhance the throughput of wireless systems [4]. To that end, when more than one subchannel is used to improve the capacity, the technique is called *multiple beamforming* [3]. In other words, multiple beamforming is a special case of *spatial multiplexing* in which SVD-based linear processing is employed at the transmitter and receiver sides [5]. In this letter, we focus on the performance analysis of single and multiple beamforming. First, we show that single beamforming achieves the maximum diversity available in space. We then calculate the diversity order for multiple beamforming. A general study of the tradeoff between diversity and multiplexing gain appeared in [6] with CSI only at the receiver.

Paper approved by N. Al-Dhahir, the Editor for Space–Time, OFDM, and Equalization of the IEEE Communications Society. Manuscript received June 14, 2005; revised October 31, 2005 and December 21, 2005. This work was supported by the Center for Pervasive Communications and Computing. This paper was presented in part at the IEEE Vehicular Technology Conference, Stockholm, Sweden, May 2005.

The authors are with the Center for Pervasive Communications and Computing, Department of Electrical Engineering and Computer Science, University of California, Irvine, CA 92697 USA (e-mail: esengul@uci.edu; eakay@uci.edu; ayanoglu@uci.edu).

Digital Object Identifier 10.1109/TCOMM.2006.876841

The rest of the letter is organized as follows. Section II gives a brief overview of the channel model and the beamforming concept. The pairwise error probability (PEP) analyses of both single and multiple beamforming are given in Section III. Simulation results supporting our analytical analysis are given in Section IV. Finally, we end the paper with a brief conclusion in Section V.

II. Channel Model and Beamforming Overview

We assume a quasi-static flat-fading MIMO channel model, where channel fading parameters are modeled as independent, identically distributed (i.i.d.) complex Gaussian random variables. Let us denote the quasi-static Rayleigh flat fading $N \times M$ MIMO channel as $\mathbf{H}$, where N is the number of transmit antennas and M is the number of receive antennas. Without any linear processing at the transmitter and the receiver, the received signal can be simply expressed as

$$\mathbf{y} = \mathbf{x}\mathbf{H} + \mathbf{n} \tag{1}$$

where $\mathbf{x}$ is a $1 \times N$ vector containing the symbols to be transmitted, and $\mathbf{n}$ is a sequence of circularly symmetric complex Gaussian noise of size $1 \times M$.

Beamforming is implemented by multiplying the symbol(s) with appropriate beamforming vector(s) both at the transmitter and the receiver. In this letter, we assume that CSI is available at both ends. In such a case, the beamforming vectors are obtained via the SVD of the channel. Then, the SVD of $\mathbf{H}$ can be written as

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H = [\mathbf{u_1 u_2} \cdots \mathbf{u_N}]\mathbf{\Lambda}[\mathbf{v_1 v_2} \ldots \mathbf{v_M}]^H \tag{2}$$

where $\mathbf{\Lambda}$ is an $N \times M$ matrix with singular values $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, in decreasing order on the main diagonal. $\mathbf{U}$ and $\mathbf{V}$ are two unitary matrices of size $N \times N$ and $M \times M$, respectively. By using SVD, the MIMO channel is divided into parallel subchannels.

In the case of multiple beamforming, multiple symbols are simultaneously sent over different parallel subchannels. The optimal vectors to be used as weights at the transmitter and receiver sides are the first S columns of $\mathbf{U}$ and $\mathbf{V}$ corresponding to the first S largest singular values of $\mathbf{H}$, when S subchannels are used simultaneously. Note that $S \leq \min(N, M)$. Then, the input–output relation for the kth subchannel for multiple beamforming becomes

$$y_k = \frac{1}{\sqrt{S}}\lambda_k x_k + n_k \tag{3}$$

where λ_k is the kth largest singular value of $\mathbf{H}$, and n_k is a complex additive white Gaussian noise (AWGN) with zero mean and variance $N_0 = 1/\text{SNR}$. When $S = 1$, multiple beamforming simply reduces to single beamforming. The elements of $\mathbf{H}$ are modeled as complex Gaussian random variables with zero mean and 0.5 variance per complex dimension. Note that the average total transmit power at the transmitter is assumed to be 1. Therefore, the received signal-to-noise ratio is SNR with the given channel and noise models.

III. Diversity Performance

A. Single Beamforming

In this section, by analyzing the PEP, we will show that single beamforming achieves the diversity order of NM for arbitrary N and M. References [7] and [8] conjecture this result, but do not give a formal proof for an arbitrary (N, M) pair. We will present an upper bound for PEP. The same result is derived in [4]. We have formerly published a tighter bound than [4] that yields this result (see [9]). However, in this letter, we will employ a method different from [9] to provide a better extension to our multiple beamforming results in Section III-B.

Single beamforming uses the subchannel with the largest gain, λ_1, to transmit only one symbol. Assume that the symbol x is sent and $\hat{x}$ is detected. Then, using the maximum-likelihood (ML) criterion, the PEP of x and $\hat{x}$, given CSI, can be written as

$$\begin{aligned} P(x \to \hat{x}|\mathbf{H}) &= P\left(|y - \lambda_1 x|^2 \geq |y - \lambda_1 \hat{x}|^2\right) \\ &= P\left(\beta - \lambda_1^2 |x - \hat{x}|^2 \geq 0\right) \\ &\leq Q\left(\sqrt{\frac{\lambda_1^2 d_{\min}^2}{2N_0}}\right) \end{aligned} \tag{4}$$

where $\beta = \lambda_1(\hat{x} - x)n^* + \lambda_1(\hat{x} - x)^* n$, $d_{\min}$ is the minimum Euclidean distance between two symbols on the constellation, and $Q(\cdot)$ is the well-known Q-function. For given $\mathbf{H}$, β is a zero-mean Gaussian random variable with variance $2N_0\lambda_1^2|x - \hat{x}|^2$. Using an upper bound for the Q function $Q(x) \leq (1/2)e^{-x^2/2}$, PEP can be bounded as

$$P(x \to \hat{x}) = E\left[P(x \to \hat{x}|\mathbf{H})\right] \leq E\left[\frac{1}{2}\exp\left(-\frac{\lambda_1^2 d_{\min}^2}{4N_0}\right)\right]. \tag{5}$$

Without loss of generality, we assume $N \leq M$. Since λ_1 is the maximum singular value, then

$$\lambda_1^2 \geq \frac{\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_N^2}{N}. \tag{6}$$

PEP can be given by

$$P(x \to \hat{x}) \leq E\left[\frac{1}{2}\exp\left(-\frac{d_{\min}^2}{4N_0 N}\sum_{i=1}^{N}\lambda_i^2\right)\right]. \tag{7}$$

Let us denote $\lambda_i^2 = \mu_i$. Note that $\{\mu_1, \mu_2, \ldots, \mu_N\}$ are the eigenvalues of $\mathbf{HH}^H$ in decreasing order [10]. The diversity order of single beamforming can be calculated using the joint probability density function (pdf) of μ_i's, which is given by [11]

$$\rho(\mu_1, \ldots, \mu_N) = K_{N,M}\prod_{i=1}^{N}\mu_i^{M-N}\prod_{i<j}(\mu_i - \mu_j)^2 e^{-\sum_{i=1}^{N}\mu_i} \tag{8}$$

where $K_{N,M}$ is a normalization constant that depends on both N and M. Using (7) and (8), PEP is upper bounded by

$$\begin{aligned} P(x \to \hat{x}) \leq \int_{\mu_N} \cdots \int_{\mu_1} \frac{1}{2} K_{N,M}\prod_{i=1}^{N}\mu_i^{M-N}\prod_{i \geq j}(\mu_i - \mu_j)^2 \\ \times e^{-\sum_{i=1}^{N}\mu_i\left(\frac{d_{\min}^2}{4N_0 N}+1\right)} d\mu_1, \ldots, d\mu_N. \end{aligned} \tag{9}$$

By simply making a change of variable, $\mu_i((d_{\min}^2/4N_0 N) + 1) \to \mu_i$, it can be shown that PEP is bounded by

$$P(x \to \hat{x}) \leq \frac{1}{2}\left(\frac{d_{\min}^2}{4N_0 N} + 1\right)^{-NM} \approx \frac{1}{2}\left(\frac{d_{\min}^2}{4N}\text{SNR}\right)^{-NM} \tag{10}$$

for high SNR. From (10), it is easy to see that the diversity order of single beamforming is NM. It is straightforward to obtain the same result for $N > M$: all N's should be replaced by M, and all M's should be replaced by N. For an alternate derivation with a tighter bound, we refer the reader to [9]. The derivation above, however, is useful for establishing a framework for our results on multiple beamforming in the next subsection.

B. Multiple Beamforming

In this section, we will show that multiple beamforming achieves the diversity order of $(N - S + 1)(M - S + 1)$ for arbitrary N, M, and S. As in the case for single beamforming, without loss of generality, we assume $N \leq M$. In multiple beamforming, multiple parallel subchannels are used for transmission of multiple symbols (i.e., multiple streams of data). However, the performance is dominated by the weakest subchannel [5]. Therefore, when S symbols are transmitted, the PEP can be bounded by

$$\begin{aligned} P(x \to \hat{x}) &\leq E\left[\frac{1}{2}\exp\left(-\frac{\mu_S d_{\min}^2}{4SN_0}\right)\right] \\ &= \int_0^\infty \frac{1}{2} e^{-\frac{\mu_S d_{\min}^2}{4SN_0}} \rho(\mu_S) d\mu_S \\ &= G(\infty) - G(0) \end{aligned} \tag{11}$$

where μ_S is the Sth largest eigenvalue of $\mathbf{HH}^H$, i.e., $\mu_S = \lambda_S^2$, $\rho(\mu_S)$ is the corresponding pdf, and where $G(\cdot)$ is the indefinite integral $\int(1/2)\exp(-(\mu_S d_{\min}^2/4SN_0))\rho(\mu_S)d\mu_S$. For systems with small N and M, one can find the marginal pdf of the Sth largest eigenvalue of $\mathbf{HH}^H$ using the joint pdf of ordered eigenvalues in (8). Then, one can analytically calculate the bounds for the PEP and diversity orders. To illustrate, we will give an example for $N = M = S = 2$ below.

Example: Diversity order of 2×2 *system (two subchannels used).* The input–output relation for each subchannel is given by (3), where $S = 2$. The diversity order of the strongest (first) subchannel is four (proved in Section III-A). The diversity order of the second subchannel can be found using (11), where expectation is taken with respect to the pdf of the second largest eigenvalue ($S=2$). The marginal pdf of μ_2 can be analytically found from (8), and can be expressed as

$$\rho(\mu_2) = 2e^{-2\mu_2}. \tag{12}$$

Using (11) and (12), PEP at high SNR can be bounded by

$$P(x \to \hat{x}) \leq \left(\frac{d_{\min}^2}{8}\mathrm{SNR}\right)^{-1}. \tag{13}$$

As seen from (13), the diversity order of the second (weakest) subchannel is one. This approach can be extended to other values of N and M. However, computational complexity to evaluate the formulas increases exponentially. In our experience, for values of $\min(N,M)$ more than four, this method is no longer practical.

To this end, we will use an approximation to the marginal pdfs to achieve analytical results for arbitrary N, M, and S. Note that in (11), the resultant PEP is highly dependent on the values of μ_S around zero. The term $G(\mu_s)$ approaches zero when $\mu_s \to \infty$, since it has an exponential factor with a negative exponent. Therefore, the pdf of the Sth largest eigenvalue around zero is essential in determining the diversity performance. In this analysis, we calculate an approximation to the ith smallest eigenvalue, pursuing an approach similar to the one in [12].

Approximation to the ith Smallest Eigenvalue: Let ν_i and $\rho(\nu_i)$ be the ith smallest eigenvalue and its pdf, respectively. By integrating (8) $N-1$ times, $\rho(\nu_i)$ can be expressed as

$$\rho(\nu_i) = \int_0^{\infty}\int_0^{\nu_N}\cdots\int_0^{\nu_{i+2}}\int_0^{\nu_i}\cdots\int_0^{\nu_2}\rho(\nu_N,\ldots,\nu_1)d\nu_1\cdots d\nu_{i-1}d\nu_{i+1}\cdots d\nu_N \tag{14}$$

where $\rho(\nu_N,\ldots,\nu_1)$ is the joint pdf of eigenvalues with ordering $\nu_N > \nu_{N-1} > \ldots > \nu_1$. Since our main concern is the marginal pdfs around zero, (14) can be further simplified, assuming that ν_i is close to the origin. For every j and k smaller than i, ν_j and ν_k will also be close to the origin. Correspondingly, $(\nu_j - \nu_k)^2$ can be approximated as ν_k^2 for $j<k<i$ and $j<i<k$. With these assumptions, the integrals from $i-1$ to N in (14) can be separately calculated and result in a constant. The integrals over ν_k's, $1 \leq k < i$, can be calculated using the fact that $\int_0^x y^n e^{-y}dy \approx x^{n+1}/(n+1)$ for small x. The final form of $\rho(\nu_i)$ can be written as

$$\rho(\nu_i) = \nu_i^k h(\nu_i)e^{-\nu_i} \tag{15}$$

where $k = i(M-N+i)-1$ and $h(\nu_i)$ is a function of ν_i, consisting of polynomials and exponential functions of ν_i such that $h(0) \neq 0$. Let us define $F(\nu_i)$ as the cumulative distribution function of ν_i. From (15), it is easy to see that the first k derivatives of $F(\nu_i)$ evaluated at $\nu_i = 0$ are zero. Therefore, neglecting the higher order terms, the Taylor expansion of $F(\nu_i)$ around the origin can be approximated as

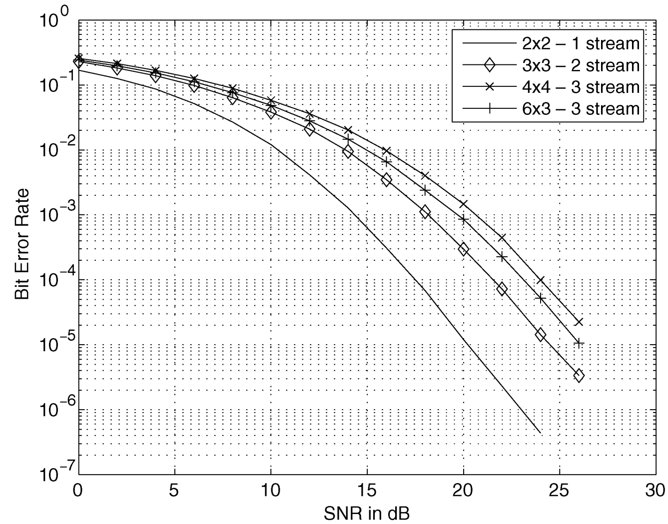

Fig. 1. Performance of single and multiple beamforming for scenario 1 with diversity order of four for flat-fading channels.

$$F(\nu_i) \approx F(0) + \alpha\nu_i^{k+1} \tag{16}$$

where $k+1 = i(M-N+i)$. To find the pdf for the ith largest eigenvalue, just a change of variable i with $N-i+1$ is needed. Then, the pdf for the Sth largest eigenvalue can be approximated as

$$\rho(\mu_S) \approx \kappa\mu_S^{(N-S+1)(M-S+1)-1} \tag{17}$$

where κ is a constant. When (17) is used in (11), the PEP for multiple beamforming can be written as

$$\begin{aligned} P(x \to \hat{x}) &\leq \gamma\left(\frac{d_{\min}^2}{4SN_0N}\right)^{-(N-S+1)(M-S+1)} \\ &= \gamma\left(\frac{d_{\min}^2}{4SN}\mathrm{SNR}\right)^{-(N-S+1)(M-S+1)} \end{aligned} \tag{18}$$

where γ is a constant. Consequently, as seen from (18), the diversity order for multiple beamforming is $(N-S+1)(M-S+1)$ when S subchannels are simultaneously used. Note that for the special case of $S=1$, diversity order reduces to NM, which was previously proved in Section III-A for single beamforming.

IV. Simulation Results

In this section, we provide Monte Carlo simulation results that quantify the analytical results derived in this letter. Figs. 1–3 show the simulation results for different antenna configurations. For all of the simulation scenarios, information bits are mapped onto 16 quadrature amplitude modulation (QAM) symbols in each subchannel. We consider packet transmission for each channel use where each packet contains 2000 information bits. To plot the curves for different scenarios, 150 000 packet transmissions are simulated for each SNR.

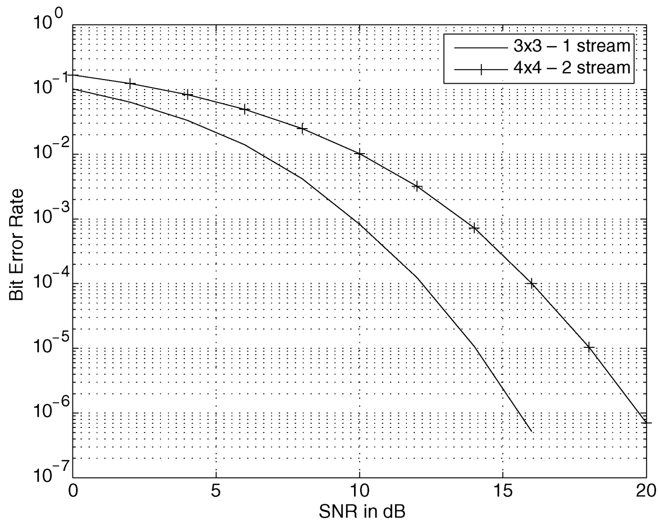

Fig. 2. Performance of single and multiple beamforming for scenario 2 with diversity order of nine for flat-fading channels.

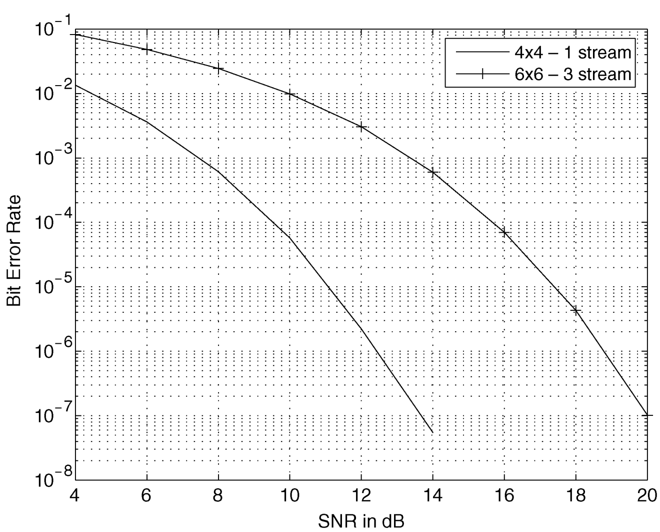

Fig. 3. Performance of single and multiple beamforming for scenario 3 with diversity order of 16 for flat-fading channels.

In Figs. 1–3, three different groups of scenarios, whose diversity orders are expected to be 4, 9, and 16, correspondingly, are presented. Note that in all groups of scenarios, there is always a curve with one stream achieving full diversity order of NM, which is also known in the literature. This will enable us to quantify the diversity orders of scenarios with $S > 1$ streams. As seen in Fig. 1, the curves for the first scenario are parallel to each other, especially for the high-SNR region. Therefore, the corresponding diversity order is four. Similar observations can be made for the other scenarios to quantify the diversity orders from the figures and the validity of analytical analysis. In the second scenario, as shown in Fig. 2, the curves for 3×3 with one stream and 4×4 with two streams are parallel in the low-error-rate region, therefore, their diversity order is nine. Finally, for the third scenario in Fig. 3, the diversity order of 4×4 with one stream and 6×6 with three streams are both 16, since they are also parallel in the low-error-rate region. Thus, simulation results verify that the diversity order of multiple beamforming with S subchannels is $(N - S + 1)(M - S + 1)$.

V. CONCLUSION

In this letter, we focused on MIMO systems with full CSI both at the transmitter and the receiver. We analyzed the PEP and marginal pdfs of channel eigenvalues. First, we showed that over Rayleigh flat-fading channels, single beamforming achieves the maximum spatial diversity order of NM. Next, we analytically showed that the diversity order of multiple beamforming is $(N - S + 1)(M - S + 1)$. We provided simulation results that verify the analytical results.

REFERENCES

[1] G. Foschini and M. J. Gans, "On limits of wireless communications in a fading environment when using multiple antennas," *Wireless Pers. Commun.*, vol. 6, no. 3, pp. 311–335, Mar. 1998.

[2] H. Jafarkhani, *Space-Time Coding: Theory and Practice*. Cambridge, U.K.: Cambridge Univ. Press, 2005.

[3] D. P. Palomar, J. M. Cioffi, and M. A. Lagunas, "Joint Tx-Rx beamforming design for multicarrier MIMO channels: A unified framework for convex optimization," *IEEE Trans. Signal Process.*, vol. 51, no. 9, pp. 2381–2401, Sep. 2003.

[4] A. Paulraj, R. Nabar, and D. Gore, *Introduction to Space-Time Wireless Communications*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[5] H. Sampath, "Linear precoding and decoding for multi-input multi-output (MIMO) wireless channels," Ph.D. dissertation, Stanford Univ., Stanford, CA, Apr. 2001.

[6] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.

[7] M. Wennstrom, M. Helin, A. R. , and T. Oberg, "On the optimality and performance of transmit and receive space diversity in MIMO channels," in *Proc. IEE Sem. MIMO: Commun. Syst. From Concept to Implementations*, 2001, vol. 4, pp. 1–6.

[8] J. Choi, "Performance analysis of the closed-loop transmit antenna diversity system over Rayleigh fading channels," *IEEE Trans. Veh. Technol.*, vol. 51, no. 4, pp. 767–771, Jul. 2002.

[9] E. Akay, E. Sengul, and E. Ayanoglu, "Performance analysis of beamforming for MIMO OFDM with BICM," in *Proc. IEEE Int. Conf. Commun.*, Seoul, Korea, May 2005, pp. 613–617.

[10] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1990.

[11] A. Edelman, "Eigenvalues and condition numbers of random matrices," Ph.D. dissertation, Mass. Inst. Technol., Cambridge, MA, 1989.

[12] A. Khoshnevis and A. Sabharwal, "On diversity and multiplexing gain of multiple antenna systems with transmitter channel information," in *Proc. Allerton Conf. Commun., Control, Comput.*, Monticello, IL, Oct. 2004, vol. 3, pp. 1571–1580.