Variable-Dilation Embeddings of Hypercubes into Star Graphs: Performance Metrics, Mapping Functions, and Routing

Marcelo Moraes de Azevedo\textsuperscript{1}, Nader Bagherzadeh\textsuperscript{1}, and Shahram Latifi\textsuperscript{2}

\textsuperscript{1} Dept. of Elec. & Comp. Engr. – Univ. of Calif. – Irvine, CA 92717
\textsuperscript{2} Dept. of Elec. & Comp. Engr. – Univ. of Nevada – Las Vegas, NV 89154

Abstract. We present load 1 embeddings of a k-dimensional hypercube $Q_k$ into an n-dimensional star graph $S_n$. Dimension $i$ links of $Q_k$ are mapped into paths of length at most $d_i$ in $S_n$, where $d_i$ varies with $i$ rather than being fixed. Our embeddings are an attractive alternative to previously known techniques, producing small average dilation and small average congestion without sacrificing expansion. We provide a thorough characterization of our embeddings, which spans several combinations of node mapping functions and routing algorithms in $S_n$.

1 Introduction

The star graph [1] is regarded as an attractive network for parallel processing, featuring smaller degree and diameter than a hypercube [5] of comparable size. However, the repertory of star graphs algorithms is still small compared to that of the hypercube. In this paper, we investigate load 1 embeddings of $Q_k$ into $S_n$, which can be used to port algorithms developed for the hypercube to a star graph. Previous work on this problem sought to minimize dilation and expansion, but has produced embeddings for which a trade-off between these metrics exists [7]. The difficulty in minimizing dilation and expansion is due to topological differences between the two networks (e.g., degree and minimum cycle length) [7]. Moreover, previous research on embeddings of $Q_k$ into $S_n$ has not addressed some important performance metrics, which are discussed in this paper. These include average dilation, congestion, and average congestion. The average dilation and average congestion metrics are good approximations for the communication slowdown induced by an embedding, and are often correlated. In particular, average dilation has been used as a standard performance metric in practical evaluations of embedding heuristics into hypercubes [4].

We present variable-dilation embeddings (VDEs) which consistently achieve small average dilation and small average congestion (e.g., one of our VDE techniques produces values for these metrics respectively in the ranges [1.50, 3.24] and [1.00, 3.21], for $n = 4 \rightarrow 10$). Simultaneously, the expansion of our VDEs matches that achieved by dilation 4 embeddings of [7]. Another advantage which stems

\textsuperscript{*} This research is supported in part by CNPQ, Brazil, under the grant No. 200392/92-1.
from our techniques include the capability of employing unused nodes in $S_n$ (up to 100% capacity) to host additional VDEs (see [2] for more on this topic).

Using several performance metrics which are defined in Sec. 2, and a combination of mathematical analysis and computer simulation, we provide a thorough characterization of our VDEs. Metrics which are derived analytically include expansion, dilation, and dilation along each of the hypercube dimensions. Average dilation, average congestion, and congestion are computed with a custom simulation program, which supports all of the VDEs presented in this paper. Measures for these last three metrics were computed over a selection of four different node mapping functions (NMFs), and four different routing algorithms in $S_n$. The paper illustrates some of the measures we obtained, pointing out the most promising combinations of NMFs and routing algorithms.

2 Variable-Dilation Embeddings (VDEs)

Performance Metrics: Definitions. Let $G_k = \{V(G_k), E(G_k)\}$ be a hierarchical $k$-dimensional graph, such that $G_{k+1}$ is obtained recursively from $G_k$ many copies of $G_k$. We refer to the links connecting the $\epsilon(k)$ copies of $G_k$ that exist within $G_{k+1}$ as dimension $(k + 1)$ links.

An embedding of $G_k$ into $H_n$, which we denote by $f : G_k \mapsto H_n$, is a mapping of $V(G_k)$ into $V(H_n)$ and of $E(G_k)$ into paths of $H_n$. In this paper, $f$ is uniquely specified by a node mapping function (NMF) $f_V : V(G_k) \mapsto V(H_n)$ and a deterministic routing algorithm $r_H$ of $H_n$. Thus, a link $(u, v)$ of $G_k$ is mapped to a path $f(u, v) = r_H(f_V(u), f_V(v))$.

The node image of $f$ is $f(V(G_k)) = \{f_V(u) : u \in V(G_k)\}$. The load of $f$, denoted by $\lambda(f)$, is the maximum number of nodes of $G_k$ that are mapped to any single node of $H_n$. The dilation of $f$ is $d(f) = \max\{\text{dist}_H(f_V(u), f_V(v)) : (u, v) \in E(G_k)\}$, where $\text{dist}_H(x, y)$ is the distance in $H_n$ between two vertices $x$ and $y$ of $H_n$. The expansion of $f$ is $X(f) = |V(H_n)|/|V(G_k)|$.

Let $E_i(G_k)$ denote the subset of dimension $i$ links in $E(G_k)$. The dilation of $f$ along the $i^{th}$ dimension of $G_k$ is $d_i(f) = \max\{\text{dist}_H(f_V(u), f_V(v)) : (u, v) \in E_i(G_k)\}$. Hence, $d_k(f) = \max\{d_i(f) : 1 \leq i \leq k\}$. $f$ is a variable-dilation embedding (VDE) if $d_i(f) < d_k(f)$, for at least one dimension $i$ of $G_k$. Accordingly, $f$ is a fixed-dilation embedding if $d_i(f) = d_k(f)$, $\forall i, 1 \leq i \leq k$. The dilation vector of $f$ is $d(f) = [d_1(f), d_2(f), \ldots, d_k(f)]$. The average dilation of $f$ is $d_{avr}(f) = \left(\sum_{(u,v) \in E(G_k)} \text{dist}_H(f_V(u), f_V(v))\right) / |E(G_k)|$.

Let $(u, v)$ and $(x, y)$ be links of $G_k$ and of $H_n$, respectively. The congestion induced by $(u, v)$ into $(x, y)$, denoted by $c_{g(x,y)}(f(u,v))$, is 1 if $f(u,v)$ traverses $(x, y)$, and 0 otherwise. The congestion induced by $f$ into $(x, y)$ is $c_{g(x,y)}(f) = \sum_{(u,v) \in E(G_k)} c_{g(x,y)}(f(u,v))$. The congestion of $f$ is $c_{g}(f) = \max\{c_{g(x,y)}(f) : (x, y) \in E(H_n)\}$. The congestion induced by dimension $i$ links of $G_k$ into $H_n$ is $c_{g}(E_i(G_k)) = \max\{\sum_{(u,v) \in E_i(G_k)} c_{g(x,y)}(f(u,v)) : (x, y) \in E(H_n)\}$. The link image of $f$ is $f(E(G_k)) = \{(x, y) \in E(H_n) : c_{g(x,y)}(f) \geq 1\}$. The average congestion of $f$ is $c_{g_{avr}}(f) = \left(\sum_{(x,y) \in f(E(G_k))} c_{g(x,y)}(f)\right) / |f(E(G_k))|^{-1}$.
The Guest Graph. A $k$-dimensional hypercube graph $Q_k$ contains $2^k$ nodes, which are labeled with binary strings of length $k$. A node $\phi = q_1 \ldots q_k$ is connected to $k$ distinct nodes, respectively labeled with strings $\phi_i = q_1 \ldots \overline{q_i} \ldots q_k$, $1 \leq i \leq k$, where $\overline{q_i}$ denotes the binary negation of bit $q_i$ [5]. The link connecting $\phi$ and $\phi_i$ is a dimension $i$ link of $Q_k$.

The Host Graph. An $n$-dimensional star graph $S_n$ contains $n!$ nodes which are labeled with the $n!$ possible permutations of $n$ distinct symbols. In this paper, we use the integers $\{1, 2, \ldots, n\}$ to label the nodes of $S_n$. A node $\pi = p_1 \ldots p_i \ldots p_n$ is connected to $(n - 1)$ distinct nodes, respectively labeled with permutations $\pi_i = p_1 \ldots p_{i-1} p_{i+1} \ldots p_n$, $2 \leq i \leq n$ [1]. The link connecting $\pi$ and $\pi_i$ is a dimension $i$ link of $S_n$.

The Intermediary Graph. Our embeddings of $Q_k$ into $S_n$ use an $(n - 1)$-dimensional mesh of size $2 \times 3 \times \ldots \times n$ as an intermediary reference graph. We denote such a graph by $M_{n-1}$, and label its nodes with $(n - 1)$-integer vectors $m_1 \ldots m_i \ldots m_{n-1}$, where $0 \leq m_i \leq i$. A node $\omega = m_1 \ldots m_i \ldots m_{n-1}$ is connected to at most $2n - 3$ distinct nodes, respectively labeled with vectors $\omega_i^- = m_1 \ldots (m_i - 1) \ldots m_{n-1}$ and $\omega_i^+ = m_1 \ldots (m_i + 1) \ldots m_{n-1}$, $1 \leq i \leq n$. $\omega_i^-$ ($\omega_i^+$) is a left (right) dimension $i$ neighbor of $\omega$ if $\omega_i^-$ ($\omega_i^+$) exists.

NMFs $g_V : V(M_{n-1}) \mapsto V(S_n)$. Our embeddings of $Q_k$ into $S_n$ use two-step NMFs. Initially, we employ an NMF $h_V : V(Q_k) \mapsto V(M_{n-1})$. The second step uses an NMF $g_V : V(M_{n-1}) \mapsto V(S_n)$. The composite NMF $f_V : V(Q_k) \mapsto V(S_n)$ is denoted by $f_V = h_V \circ g_V$.

In what follows, we describe four different NMFs $g_V : V(M_{n-1}) \mapsto V(S_n)$, which we denote by $g^{\text{nonh}}_V$, $g^{\text{monh}}_V$, $g^{\text{hier}}_V$, and $g^{\text{ghier}}_V$. These NMFs are respectively referred to as the non-hierarchical NMF [6, 8], the modified non-hierarchical NMF [2], the hierarchical NMF, and the quasi-hierarchical NMF. All four NMFs embed $M_{n-1}$ into $S_n$ with load 1, expansion 1, and dilation 3. The dilation vectors of the embeddings produced by $g^{\text{nonh}}_V$, $g^{\text{monh}}_V$, $g^{\text{hier}}_V$, and $g^{\text{ghier}}_V$ are respectively $[3, \ldots, 3, 1]$, $[3, \ldots, 3]$, $[1, 2, 3, \ldots, 3]$, and $[3, 2, 3, \ldots, 3]$.

Let $\pi$ be a permutation of $n$ symbols. We denote the transposition of symbols $i$ and $j$ in $\pi$ by $(i \ j)$. Similarly, we denote the transposition of the symbols occupying the $i$th and the $j$th positions in $\pi$ by $(i \ j)_p$. We define an operator $\circ$ which applies transpositions to permutations. Hence, $2134 \circ (2 \ 4)_p = 4213$, and $2134 \circ (2 \ 4)_p = 2314$. NMFs $g^{\text{nonh}}_V$, $g^{\text{monh}}_V$, $g^{\text{hier}}_V$, and $g^{\text{ghier}}_V$ can be generically described by the algorithm depicted in Table 1. For example, $g^{\text{monh}}_V(102) = 2134 \circ (3 \ 2)_p \circ (3 \ 2)_p = 2314$.

NMF $h_V : V(Q_k) \mapsto V(M_{n-1})$. In what follows, we present an NMF $h_V : V(Q_k) \mapsto V(M_{n-1})$, which is common to all of our VDEs. We denote the corresponding composite NMFs $f_V : V(Q_k) \mapsto V(S_n)$ by $f^{\text{nonh}}_V = h_V \circ g^{\text{nonh}}_V$, $f^{\text{monh}}_V = h_V \circ g^{\text{monh}}_V$, $f^{\text{hier}}_V = h_V \circ g^{\text{hier}}_V$, and $f^{\text{ghier}}_V = h_V \circ g^{\text{ghier}}_V$.

Let $F(x, y) = x(y + 1) - 2^{x+1} + 2$, and let $n$, $\ell$, and $k$ be integers such that $n \geq 4$, $2 \leq \ell \leq \lfloor \log_2 n \rfloor$, and $F(\ell - 1, n) < k \leq F(\ell, n)$. Let $\phi[]$ and $\omega[]$ be nodes of $Q_k$ and $M_{n-1}$, respectively. An algorithmic description of $h_V$ follows:

Table 1. Algorithmic description for NMFs $g_V : V(M_{n-1}) \xrightarrow{\phi} V(S_n)$

<table>
<thead>
<tr>
<th>NMF</th>
<th>$m_1 m_2 = 00$</th>
<th>$m_1 m_2 = 10$</th>
<th>$m_1 m_2 = 11$</th>
<th>$m_1 m_2 = 12$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_V^{(r)}$</td>
<td>12345 ... 8</td>
<td>12453 ... 8</td>
<td>13425 ... 8</td>
<td>14235 ... 8</td>
</tr>
<tr>
<td>$g_V^{(r)}$, $g_V^{(h)}$</td>
<td>12345 ... 8</td>
<td>12453 ... 8</td>
<td>13425 ... 8</td>
<td>14235 ... 8</td>
</tr>
<tr>
<td>$g_V^{(h)}$, $g_V^{(h)}$</td>
<td>12345 ... 8</td>
<td>12453 ... 8</td>
<td>13425 ... 8</td>
<td>14235 ... 8</td>
</tr>
</tbody>
</table>

For $i : 3 \mapsto (n - 1)$, apply the first $m_i$ transpositions specified for dimension $i$ and NMF $g_V$ in Table 1b to $\pi_0$.

<table>
<thead>
<tr>
<th>NMF</th>
<th>$i = 3$</th>
<th>$i = 4$</th>
<th>$i = n - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_V^{(r)}$, $g_V^{(h)}$</td>
<td>(1 2) o (2 3) o (3 4)</td>
<td>(1 2) o ... o (4 5) ...</td>
<td>(1 2) o ... o (n - 1)</td>
</tr>
<tr>
<td>$g_V^{(r)}$, $g_V^{(h)}$</td>
<td>(4 3) o (3 2) o (2 1)</td>
<td>(5 4) o ... o (2 1)</td>
<td>(n - 1) o ... o (2 1)</td>
</tr>
<tr>
<td>$g_V^{(h)}$, $g_V^{(h)}$</td>
<td>(4 3) o (3 2) o (2 1)</td>
<td>(5 4) o ... o (2 1)</td>
<td>(n - 1) o ... o (2 1)</td>
</tr>
</tbody>
</table>

for $(i = 1; i < n; i + 1)$, $\omega[i] = 0$;
for $(i = 1; F(e - 1, n) < k; e = c + 1)$
for $(i = F(e - 1, n) + 1; i \leq \min(F(e, n), k); i = i + 1)$
$\omega[i - F(e - 1, n) + 2^e - 2] = \omega[i - F(e - 1, n) + 2^e - 2] + 2^{e-1} \cdot \phi[i]$;

Table 2 depicts the VDE $h : Q(4) \xrightarrow{\phi} M(3)$ produced by NMF $h_V$. Properties for $h$ are $X(h) = 1.5$, $A(h) = 1$, $d(h) = 2$, $d(h) = [1, 1, 1, 2]$, and $d_{av}(h) = 1.25$. Also shown in Table 2 are VDEs $f : Q(4) \xrightarrow{\phi} S(4)$ produced by $f_V^{(h)}$, $f_V^{(h)}$, $f_V^{(r)}$ and $f_V^{(h)}$. The properties of $f^{(h)}$, for example, are $X(f^{(h)}) = 1.5$, $A(f^{(h)}) = 1$, $d(f^{(h)}) = 4$, $d_{av}(f^{(h)}) = [1, 2, 3, 4]$, and $d_{av}(f^{(h)}) = 2$.

Table 2. Image nodes for NMFs $h_V$, $f_V^{(h)}$, $f_V^{(h)}$, $f_V^{(h)}$, and $f_V^{(h)}$ (n = k = 4)

<table>
<thead>
<tr>
<th>NMF</th>
<th>Hypercube node ($\phi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_V$</td>
<td>0000 0000 0000 0000 0001 0001 0001 0001 0010 0010 0010 0010 1100 1100 1100 1100 1101 1101 1101 1101 1110 1110 1110 1110</td>
</tr>
<tr>
<td>$f_V^{(h)}$</td>
<td>1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234</td>
</tr>
<tr>
<td>$f_V^{(h)}$</td>
<td>1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234</td>
</tr>
<tr>
<td>$f_V^{(h)}$</td>
<td>1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234</td>
</tr>
<tr>
<td>$f_V^{(h)}$</td>
<td>1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234 1234</td>
</tr>
</tbody>
</table>

Properties of VDEs $f : Q_k \xrightarrow{\phi} S_n$. (see [3] for a proof).

**Theorem 1.** Let $F(x, y) = x(y + 1) - 2^{x+1} + 2$, and let $n$, $\ell$, and $k$ be integers such that $n \geq 4$, $2 \leq \ell \leq \log_2 n$, and $F(\ell - 1, n) < k \leq F(\ell, n)$. Let $f_V$ be one of the node mapping functions $f_V^{(h)}$, $f_V^{(h)}$, $f_V^{(h)}$, $f_V^{(h)}$, and $f_V^{(h)}$, and let $f : Q_k \xrightarrow{\phi} S_n$ be one of the corresponding embeddings $f_V^{(h)}$, $f_V^{(h)}$, $f_V^{(h)}$, $f_V^{(h)}$, and $f_V^{(h)}$ generated by $f_V$. For each $f$, we define integers $\gamma_i(f)$ as follows:

- \( \gamma_i(f^{mon,h}) = 0 \) if \( i = F(e, n) \), \( \forall e \in [1, \ell] \), and \( \gamma_i(f^{mon,h}) = 2 \) otherwise.
- \( \gamma_i(f^{pmo,h}) = 2 \), \( \forall i \).
- \( \gamma_i(f^{hier}) = 0 \), \( \gamma_i(f^{hier}) = 1 \), and \( \gamma_i(f^{hier}) = 2 \), for all \( i > 2 \).
- \( \gamma_i(f^{hier}) = 0 \), \( \gamma_i(f^{hier}) = 1 \), and \( \gamma_i(f^{hier}) = 2 \), for all \( i > 2 \).

Then, \( \lambda(f) = 1 \), \( X(f) = n/l2^k \), \( d(f) = \max_{i \in \{d(f_i)\}} \), and:

\[
D(f) = \frac{1 + \gamma_1(f) + 2 + \gamma_2(f) + \cdots + 2^{n-1} + \gamma_i(f) + \cdots + 2^{n-1} + \gamma_n(f)}{n-1}.
\]

**Routing and Simulation Results.** We used simulation to characterize other important metrics of our VDEs, such as average dilation, average congestion, and congestion induced by dimension \( i \) links of \( Q_k \). The tool supports all of the NMFs \( f^{mon,h} \), \( f^{pmo,h} \), \( f^{hier} \), and \( f^{hier} \), and four deterministic routing algorithms in \( S_n \). These algorithms are denoted by \( r^{can}_S \), \( r^{can}_S \), \( r^{can}_S \), and \( r^{can}_S \).

Recall that a link \((u, v) \) is an edge of \( Q_k \) is mapped to a path \( f(u, v) \) in \( S_n \) with endpoints \( f_L(u) \) and \( f_L(v) \). Let \( \sigma_{u,v} \) denote the permutation which sorts \( f_L(u) \) into \( f_L(v) \). Each routing algorithm employs a particular format for the cyclic representation of \( \sigma_{u,v} \), and executes \( \pi_{u,v} \) accordingly [3]. \( r^{can}_S \), \( r^{can}_S \), \( r^{can}_S \), and \( r^{can}_S \) represent \( \pi_{u,v} \) in canonical format, reverse canonical format, even-only canonical format, and odd-only canonical format, respectively [3].

Figure 1 depicts the average dilation and the average congestion produced by VDEs employing the combination \( f^{hier} + r^{can}_S \). Similar plots for other combinations of NMF and routing algorithm can be found in [3]. We consider the cases \( k : 2 \rightarrow 19 \) and \( n : 4 \rightarrow 10 \), which correspond respectively to hypercubes of sizes \( 4 \rightarrow 524,288 \), and star graphs of sizes \( 24 \rightarrow 3,628,800 \). \( f^{mon,h}, f^{pmo,h}, f^{hier}, \) and \( f^{hier} \) achieve average dilation which lie in the ranges [2,25,3,33], [2,00,3,60], [1,50,3,21], and [1,50,3,24], respectively. Accordingly, measures for average congestion produced by the combinations \( f^{mon,h} + r^{can}_S, f^{pmo,h} + r^{can}_S, f^{hier} + r^{can}_S, f^{hier} + r^{can}_S, \) and \( f^{hier} + r^{can}_S \) lie in the ranges [1,00,5,47], [1,19,2,96], [1,07,3,81], and [1,00,3,21], respectively. Note that the average dilation computed for a VDE does not depend on the choice of routing algorithm.

Hierarchical NMFs achieve smaller average dilation than non-hierarchical NMFs, and produce smaller average congestion when used in combination with \( r^{can}_S \). Conversely, \( r^{can}_S \) minimizes the average congestion produced by non-hierarchical NMFs. The combinations \( f^{mon,h} + r^{can}_S \) and \( f^{hier} + r^{can}_S \) seem to be the most appropriate for star graphs employing canonical and reverse canonical routing, respectively. \( r^{can}_S \) and \( r^{can}_S \) produce congestion metrics which lie between the minima and maxima obtained with \( r^{can}(r^{can}) \) and \( r^{can}(r^{can}) \), when used in combination with hierarchical and non-hierarchical NMFs, respectively.

Several of our VDEs produce congestion 1 or 2 on the links of \( S_n \) when a single dimension of \( Q_k \) is used [3]. This is particularly important for algorithms which employ only a fraction of the links of \( Q_k \) at any point of their execution (e.g., SIMD algorithms). From the viewpoint of congestion, some interesting results are: 1) \( f^{mon,h} + r^{can}_S \) produces VDEs whose congestion is less than \( k \), over the ranges \( k : 2 \rightarrow 19 \) and \( n : 4 \rightarrow 10 \), and 2) \( f^{hier} + r^{can}_S \) produces VDEs with congestion 1 when \( k \leq n - 1 \) [3].
3 Conclusion

This paper presented novel techniques for embedding a hypercube into a star graph. Our embeddings are designed for performance, and consistently produce small average dilation and small average congestion. We achieve these goals by employing variable-dilation embeddings, and a careful selection of node mapping functions and routing algorithms. Our techniques demonstrated the possibility of embedding large hypercubes into the star graph, with corresponding small expansion. On continued research, we are expanding our investigation on congestion metrics to a related technique introduced by the authors. Such a technique is referred to as packing, and can produce optimal expansion (i.e., 1) [2].

References


This article was processed using the \LaTeX\ macro package with LLNCS style.