EXTENSIONS TO COMPLEX MATERIALS OF THE FITZGERALD MODEL FOR THE SOLUTION OF ELECTROMAGNETIC PROBLEMS

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ABSTRACT
Electromagnetic phenomena can be simulated by the dynamics of a mechanical system as long as the Hamiltonian of the electromagnetic and the mechanical systems coincide. In this paper we present a generalization of G.F. FitzGerald's pulleys and rubber-bands mechanical model for the interaction of electromagnetic waves with complex media. We show a direct analogy between the FitzGerald model and the electric vector potential formulation, at each stage of the extension of the original model: each mechanical observable has a unique correspondence in the vector potential formulation. This strict analogy allows further inductive developments of the mechanical model and extends the pedagogical importance of the original FitzGerald model. As a consequence very complex materials from the electromagnetic point of view, such as frequency dependent magneto dielectric materials are easily understood and implemented with simple modifications in the mechanical system. The condense node representation of the field in the vector potential formulation results in lower grid dispersion compared to other numerical techniques such as the Finite Difference Time Domain (FDTD). We describe several applications, such as classical scattering problems from dielectric, magnetically permeable, dielectrically lossy and Debye materials. The simulations are validated with comparison to canonical solutions, or with FDTD calculations.

1. INTRODUCTION
Mechanical analog models can provide an excellent time domain visualization tool for propagation, scattering and radiation of electromagnetic waves in dispersive media. It has been known since the late 19th century that electromagnetic phenomena can be simulated by the dynamics of a mechanical system, as long as the Hamiltonians of the mechanical and the electromagnetic systems coincide. It was shown by Diaz [1, 2] that George Francis FitzGerald's 1885 [3] model of electromagnetic propagation leads to a finite difference numerical formulation that is different from the conventional Finite Difference Time Domain method (FDTD). This is because FDTD is based upon the discretization of Maxwell's equations in the classical formulation, while the Diaz time domain discretization of FitzGerald's mechanical model coincides with the discretization of Maxwell's equations in the vector potential formulation [4]. In this paper we present in

Received 1 September 1997; accepted 20 November 1997.

Electromagnetics, 18:35–65, 1998  
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0272-6343/98 $12.00 + .00
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Electromagnetic phenomena can be simulated by the dynamics of a mechanical system as long as the Hamiltonian of the electromagnetic and the mechanical systems coincide. In this paper we present a generalization of G.F. FitzGerald's pulleys and rubber-bands mechanical model for the interaction of electromagnetic waves with complex media. We show a direct analogy between the FitzGerald model and the electric vector potential formulation, at each stage of the extension of the original model: each mechanical observable has a unique correspondence in the vector potential formulation. This strict analogy allows further inductive developments of the mechanical model and extends the pedagogical importance of the original FitzGerald model. As a consequence very complex materials from the electromagnetic point of view, such as frequency dependent magneto dielectric materials are easily understood and implemented with simple modifications in the mechanical system. The condense node representation of the field in the vector potential formulation results in lower grid dispersion compared to other numerical techniques such as the Finite Difference Time Domain (FDTD). We describe several applications, such as classical scattering problems from dielectric, magnetically permeable, dielectrically lossy and Debye materials. The simulations are validated with comparison to canonical solutions, or with FDTD calculations.

1. INTRODUCTION
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including scattering from a cylinder, an eigenvalue problem, and an echo experiment. All extensions of the model are continuously supported by analogies with the vector potential formulation (see Appendices). We also show how the condense node representation of the field causes lower grid dispersion with respect to the traditional Yee scheme of FDTD. For the two dimensional cases considered, our time domain techniques requires 38% less memory and, consequently, 38% less computational time.

2. THE FITZGERALD PULLEY AND RUBBER-BAND MODEL
Consider the distributed discrete system of Figure 1. Pulleys of moment of inertia $I_{i,j}$ are connected to each other through rubber bands of elasticity $k_{i+1/2,j+1/2}$.

![Fig.1 Array of rigid pulleys connected by rubber bands and corresponding equivalent electric quantities.](image1)

If at time $t=0$ pulley $(i,j)$ is spun (Figure 2a), its rotation strains the four rubber bands connecting it to its four neighbors.(Figure 2b). The total force of tension and compression applied at the peripheries of the pulleys exert to each one an angular acceleration $\alpha$ through Newton's second law: their angular velocity $\omega$ increases by $\alpha dt$ after a time step $dt$ (Figure 2c).

![Fig.2 Action-reaction mechanism of propagation of motion; the dark gray represents the spanned angle due to the initial action (a) and consequent reaction (c) ](image2)

Thus, an angular velocity pulse applied to the central pulley propagates outwards by action and reaction to all pulleys of the system. FitzGerald showed that this behavior mimics exactly generation and propagation of electromagnetic waves in Maxwell's curl equations: the inertia of the pulley $I$ represents the medium permeability ($\mu$), and the
elastic constant $k$ represents the medium inverse permittivity ($1/e$). FitzGerald further noted that if the rubber bands are allowed to slip then we obtain a lossy dielectric medium with the conductivity being inversely proportional to the coefficient of friction between the rubber bands and the pulley. Of course, by Heaviside’s *duplex equations* [5] we can also choose to identify $I=\varepsilon k=1/\mu$ according to Diaz [2]. Application of Greenspan’s approach [6] to this mechanical system yields a set of finite difference equations as follows. Consider for the sake of simplicity the one dimensional system as shown in Fig. 3.

Fig. 3 Pulley and rubber-bands represented as springs; when the i-th pulley is spun clockwise the four neighboring springs react with the restoring forces.

Note that the elasticity of the rubber band is represented pictorially as a spring with elastic constant $k$. Each pulley is connected to two springs as illustrated in Figure 3, which contribute to the overall torque acting on it. The generic i-th pulley is subject to four forces due to the neighboring rubber bands:

\[
\begin{align*}
F_i^{(1)} &= -k_i^{(1)}(\theta_i - \theta_{i+1})a \\
F_i^{(2)} &= -k_i^{(2)}(\theta_i - \theta_{i+1})a \\
F_{i+1}^{(1)} &= -k_{i+1}^{(1)}(\theta_i - \theta_{i+1})a \\
F_{i+1}^{(2)} &= -k_{i+1}^{(2)}(\theta_i - \theta_{i+1})a
\end{align*}
\]  

(1)

As a consequence the resulting torque $T_i$ acting on the i-th pulley, is given by

\[
T_i = (F_i^{(1)} + F_i^{(2)})a - (F_{i+1}^{(2)} + F_{i+1}^{(1)})a = k_i a^2 (\theta_i - \theta_{i+1}) + k_{i+1} a^2 (\theta_i - \theta_{i+1}),
\]

(2)

see Fig. 4. The acceleration of the i-th pulley after a time step $\Delta t$ is equal to:

\[
\frac{\Delta \omega}{\Delta t} = \frac{k_i a^2}{I} (\theta_i - \theta_{i+1}) + \frac{k_{i+1} a^2}{I} (\theta_i - \theta_{i+1}),
\]

(3)

$\omega$ being the angular velocity at time step $n$. If we now identify the diameter of the pulley with the grid size, $\Delta x=a$, previous expression becomes after rearrangement.
Fig. 4 Torque resulting from the composition of the four forces, acting on the pulley of radius $a$, width $h$ and density $\rho$

\[
\left( \frac{\Delta \omega}{\Delta t} \right)_i = \left( \frac{\Delta \alpha}{\Delta x} \right)_i \frac{I_i}{\frac{\theta_i^0 - \theta_{i+1}^0}{\Delta x} - \frac{\theta_i^+ - \theta_{i-1}^+}{\Delta x}} \]

(4)

since the inertia of the pulley $I_i$ with radius $a$, width $h$ and density $\rho$ can be written as

\[
I_i = \frac{1}{2} ma^2 = \frac{1}{2} \left( \frac{\pi a^3 h}{\rho} \right) a^2 = \frac{\pi h a^4}{2 \rho} = \frac{\pi h}{2 \rho} (\Delta x)^4 \propto (\Delta x)^4
\]

(5)

If we take the limit for $\Delta t \to 0$ and $\Delta x \to 0$, since $I$ is proportional to $(\Delta x)^4$ the continuum differential equation of (4) together with the definition of angular velocity becomes:

\[
\begin{aligned}
\frac{\partial \omega}{\partial t} &= \frac{a^2}{I_i} \frac{\partial}{\partial x} \left( k a^2 \frac{\partial}{\partial x} \right) \\
\frac{\partial \theta}{\partial t} &= \omega
\end{aligned}
\]

(6)

The electrical analog of the same one dimensional Maxwell's curl equations is given in great detail in Appendix A, and it leads to [7]:

\[
\begin{aligned}
\frac{\partial E}{\partial t} &= \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial A}{\partial x} \right) \\
\frac{\partial A}{\partial t} &= -E_i
\end{aligned}
\]

(7)

Note that in the derivation of eq. (7) there is no constraint on the space invariance of the dielectric constant nor of the magnetic permeability. The extension to the two dimensional case is straightforward and is omitted. A summary of the correspondence between the mechanical, see eq.6, and the electrical quantities, see eq.7, is presented in Table I. In the mechanical model the presence of a dielectric medium ($\varepsilon \neq \varepsilon_0$) is taken
into account by increasing the moment of inertia of the pulley; while the presence of a
magnetically permeable medium (\(\mu \neq \mu_0\)) is modeled by modifying the elasticity of the
rubber (the elastic constant of the spring ) \([8, 9]\).

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_z)</td>
<td>(z)-comp. electric field</td>
</tr>
<tr>
<td>(dE_z/dt)</td>
<td>(E) rate of change</td>
</tr>
<tr>
<td>(\dot{\alpha})</td>
<td>(\alpha) angular acceleration</td>
</tr>
<tr>
<td>(\dot{\varepsilon})</td>
<td>(\varepsilon) permeability</td>
</tr>
<tr>
<td>(\mu)</td>
<td>(1/(k a^2)) rubber elasticity</td>
</tr>
<tr>
<td>(\Delta x)</td>
<td>(a) pulley radius</td>
</tr>
</tbody>
</table>

Table 1 Equivalence between electrical and mechanical quantities in our model.

3. EXTENSION TO DIELECTRIC LOSSY MATERIALS
Our first extension of FitzGerald's model to mimic dielectric lossy materials consists of
the immersion of one or more pulleys in a bath of viscous fluid as shown in Figure (5).

\[
T_{\text{effective}} = T_{\text{load}} - \xi \omega, \tag{8}
\]

which translates into the equation for the acceleration

\[
\frac{\partial \omega}{\partial t} = \frac{a^2}{I} \frac{\partial}{\partial \chi} \left( k \frac{\partial^2 \theta}{\partial \chi^2} \right) - \frac{a^2}{I} \xi \omega. \tag{9}
\]

It is shown in detail (see Appendix A) how to derive the vector potential equations for the
case of electric lossy materials starting from Maxwell's equations; we report here only the
final expression for the time derivative of the electric field in one dimension:
$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial A}{\partial x} \right) - \frac{\sigma}{\varepsilon} E. \quad (10)$$

By comparing eq. (9) to eq. (10) we conclude that the mechanical quantity "viscous friction coefficient" $\xi$ corresponds to the conductivity $\sigma$:

$$\xi \leftrightarrow \sigma. \quad (11)$$

4. EXTENSION TO DEBYE DIELECTRIC MATERIALS

In this section we focus our attention on dispersive media whose behavior can be modeled by a sum of Debye terms [10]. For a material characterized by a single Debye relaxation we can write [11, 12]

$$\mathbf{D} = \varepsilon \cdot \mathbf{E} + \mathbf{P}, \quad (12)$$

where

$$\frac{\partial \mathbf{P}}{\partial t} = \frac{\varepsilon - \varepsilon_s}{\tau} \mathbf{E} - \frac{1}{\tau} \mathbf{P}. \quad (13)$$

In eq. (13) $\varepsilon$ and $\varepsilon_s$ are the dielectric constants of zero (static) and "infinite" frequency, and $\tau$ is the relaxation time constant. The modification of the mechanical model takes the form of an additional weighted ring (of moment of inertia $I$) resting on the pulley and connected to it through a coefficient of friction ($Q$) as shown in Figure 6.

![Mechanical model for single electrical Debye materials.](Fig.6)

The time constant of the Debye pole $\tau$ corresponds to the ratio $I^{(s)}/Q^{(s)}$. Action and reaction of the top pulley then simulates the storage and dissipation of polarization. The new set of equations for this system can be derived from the original equations in the one dimensional case, by adding an inertial reaction due to the top pulley which is coupled to the bottom one through a friction coefficient.
Here $\omega_i^{(1)}$ represents the angular velocity of the weighted ring, $I_i^{(1)}$ the moment of inertia and $Q_i^{(1)}$, the friction coefficient. The system of equations (14) can be rewritten as:

\[
\begin{aligned}
\frac{\partial \omega_i^{(1)}}{\partial t} &= \frac{a^2}{I_i^{(1)}} \left( \omega_j - \omega_i^{(1)} \right) \\
\frac{\partial \omega_i}{\partial t} &= \frac{a^2}{I_i} T_{\text{wind}} - \frac{I_i^{(1)}}{I_i} \frac{\partial \omega_i^{(1)}}{\partial t} \\
\frac{\partial \theta_i}{\partial t} &= \omega_i \\
\end{aligned}
\]  

(14)

This extension of the FitzGerald model finds as exact analogy in the vector potential formulation of Maxwell's Equations. In this case the polarization vector needs to be introduced (we carry out a detailed derivation in Appendix B). Here we just report the final equations in the one dimensional case:

\[
\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{\varepsilon_+ - \varepsilon_-}{\tau} E_x - \frac{1}{\tau} P, \\
\frac{\partial E_x}{\partial t} &= \frac{1}{\varepsilon_+} \frac{\partial A}{\partial x} + \frac{\varepsilon_+ - \varepsilon_-}{\tau \varepsilon_+} \left( \frac{P}{\varepsilon_+ - \varepsilon_-} - E_x \right), \\
\frac{\partial A}{\partial t} &= -E_x. \\
\end{aligned}
\]  

(16)

By comparing eq.(15) to eq.(16) it is clear that there are several other analogies between the mechanical quantities and the electrical ones which we list below. The extension to double Debye materials is straightforward, in fact we can mimic double Debye terms by adding two rings surmounting the original pulley arranged concentrically as in Figure 7. Since the two rings are directly connected to the bottom pulley, they are independently coupled to it though two different friction coefficients.
Table II Equivalence between electrical and mechanical quantities in our model.

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_z/(e_s - e_{\infty})$</td>
<td>$\omega^{(1)}$</td>
</tr>
<tr>
<td>$(e_s - e_{\infty})/\tau$</td>
<td>$Q^{(1)}$</td>
</tr>
<tr>
<td>$e_s - e_{\infty}$</td>
<td>$I^{(1)}/a^2$</td>
</tr>
<tr>
<td>$e_{\infty}$</td>
<td>$1/a^2$</td>
</tr>
</tbody>
</table>

Fig.7 Coaxial arrangement of two Debye terms

The mechanics of the system is enriched by two inertial reaction contributions due to the action and reaction of the top two rings:

\[
\begin{align*}
\frac{\partial \omega^{(2)}}{\partial t} &= \frac{a^2 Q^{(2)}}{I^{(2)}}(\omega_1 - \omega^{(2)}_1) \\
\frac{\partial \omega^{(1)}}{\partial t} &= \frac{a^2 Q^{(1)}}{I^{(1)}}(\omega_2 - \omega^{(1)}_2) \\
\frac{\partial \omega_i}{\partial t} &= \frac{a^2}{I_i} M_{\text{band}} - \frac{a^2 Q^{(1)}}{I_i}(\omega_1 - \omega^{(1)}_1) - \frac{a^2 Q^{(2)}}{I_i}(\omega_2 - \omega^{(2)}_2) \\
\frac{\partial \theta}{\partial t} &= \omega_i
\end{align*}
\]

where the mechanical quantities referring to the two rings are indicated with the superscript 1 and 2 respectively. It is useful to define a weighted average of the angular velocities of the two rings, where the weighting factors are related to the relative moment of inertia of the two rings:
Therefore the average angular acceleration becomes, with the aid of eq. (17):

\[ \frac{\partial \ddot{\omega}}{\partial t} = \frac{a^2 Q^{(1)}}{I^{(1)} + I^{(2)}} (\omega - \omega^{(1)}) + \frac{a^2 Q^{(2)}}{I^{(1)} + I^{(2)}} (\omega - \omega^{(2)}) \]  

(19)

System (17) can be rewritten in the form:

\[ \begin{align*}
\frac{\partial \ddot{\omega}}{\partial t} &= \frac{a^2 Q^{(1)}}{I^{(1)} + I^{(2)}} (\omega - \omega^{(1)}) + \frac{a^2 Q^{(2)}}{I^{(1)} + I^{(2)}} (\omega - \omega^{(2)}) \\
\frac{\partial \omega}{\partial t} &= \frac{a^2}{T_{\text{total}}} \left( \frac{1}{I^{(1)} + I^{(2)}} \right) \frac{\partial \ddot{\omega}}{\partial t} \\
\frac{\partial \theta}{\partial t} &= \omega
\end{align*} \]  

(20)

The analogy with the vector potential formulation is possible also in this case, where we have defined two partial polarizations \( P_z^{(1)} \) and \( P_z^{(2)} \) and an "effective" polarization \( P_z = P_z^{(1)} + P_z^{(2)} \). Each polarization obeys the differential equation

\[ \frac{dP_z^{(n)}}{dt} = \varepsilon_\tau \frac{\partial P_z^{(n)}}{\partial t} = \varepsilon_\tau g_n E_z, \]  

(21)

so the total effect is simply given by the superposition principle, and we used the notation \( \varepsilon_\tau = \varepsilon_\tau' - \varepsilon_\tau' \), \( g_n \) and \( \tau_n \) being the weight of each pole and its time constant, respectively [10]. The equations for the vector potential formulation for double Debye materials become:

\[ \begin{align*}
\frac{1}{\varepsilon_\tau} \frac{\partial P_z^{(1)}}{\partial t} &= -g_1 \left( \frac{P_z^{(1)}}{\varepsilon_\tau} - E_z \right) - g_2 \left( \frac{P_z^{(2)}}{\varepsilon_\tau} - E_z \right) \\
\frac{\partial E_z}{\partial t} &= -\frac{1}{\varepsilon_\tau} \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial \lambda}{\partial x} \right) - \frac{1}{\varepsilon_\tau} \frac{\partial P_z^{(1)}}{\partial t} \\
\frac{\partial \lambda}{\partial t} &= -E_z
\end{align*} \]  

(22)

Each polarization corresponds to the angular velocity of one of the rings, while the total polarization corresponds to the "average" angular velocity of the rings. The analogies between the mechanical system and the electrical formulation are easily drawn and are collected in Table III. Extension to n-pole Debye materials at this point can be carried out by induction. The representation of the system is a collection of \( n \) rings placed on top of a pulley. Each ring is characterized by its own moment of inertia (\( I^{(n)} \)) and is coupled to
the bottom pulley though a friction coefficient \( Q^{(n)} \) independent of the other rings. Therefore we identify in the model the \( n \) angular velocities of the rings with the \( n \) partial polarizations [13]. The relations between the mechanical and electrostatic quantities are collected in Table IV.

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_z/(\varepsilon_s-\varepsilon_{\infty}) )</td>
<td>( \bar{\omega} )</td>
</tr>
<tr>
<td>( P_z^{(1)}/\varepsilon_s-\varepsilon_{\infty}) )</td>
<td>( \omega^{(1)} )</td>
</tr>
<tr>
<td>( P_z^{(2)}/\varepsilon_s-\varepsilon_{\infty}) )</td>
<td>( \omega^{(2)} )</td>
</tr>
<tr>
<td>( \varepsilon_1 (\varepsilon_s-\varepsilon_{\infty})/\tau_1 )</td>
<td>( Q^{(1)} )</td>
</tr>
<tr>
<td>( \varepsilon_2 (\varepsilon_s-\varepsilon_{\infty})/\tau_2 )</td>
<td>( Q^{(2)} )</td>
</tr>
<tr>
<td>( \varepsilon_{\infty} )</td>
<td>( l^{(1)}/a^2 )</td>
</tr>
<tr>
<td>( \varepsilon_{\infty} )</td>
<td>( l^{(2)}/a^2 )</td>
</tr>
</tbody>
</table>

Table III Equivalence between electrical and mechanical quantities in our model.

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_z/(\varepsilon_s-\varepsilon_{\infty}) )</td>
<td>( \bar{\omega} )</td>
</tr>
<tr>
<td>( P_z^{(n)}/g_n(\varepsilon_s-\varepsilon_{\infty}) )</td>
<td>( \omega^{(n)} )</td>
</tr>
<tr>
<td>( g_n(\varepsilon_s-\varepsilon_{\infty})/\tau_n )</td>
<td>( Q^{(n)} )</td>
</tr>
<tr>
<td>( g_n(\varepsilon_s-\varepsilon_{\infty}) )</td>
<td>( l^{(n)} )</td>
</tr>
<tr>
<td>( \varepsilon_{\infty} )</td>
<td>( l )</td>
</tr>
</tbody>
</table>

Table IV Equivalence between electrical and mechanical quantities in our model.

5. EXTENSION TO FREQUENCY DEPENDENT MAGNETIC MATERIALS

Less common but still very interesting from both theoretical and engineering point of view are dispersive magnetic materials. We again restrict out attention to materials which can be modeled by a sum of Debye terms. A single Debye magnetic medium is characterized by the following equation:

\[
B = \mu_s H + M ,
\]

(23)

where

\[
\frac{\partial M}{\partial t} = \frac{\mu_s - \mu_-}{\tau} H - \frac{1}{\tau} M .
\]

(24)

Here \( \mu_s \) and \( \mu_- \) are the permeability at zero (static) and "infinite" frequency and \( \tau \) is the relaxation time constants. In this case the mechanical model can be extended once more to describe such materials by modifying the rubber bands elasticity to attribute their elastic
constants a Debye character. The modification takes the form of an additional spring connected in series with the original one. The new spring is connected with a device immersed in a viscous fluid, so that the oscillation of the second spring is damped according to the viscous coefficient of the fluid $\gamma$, see Figure 8.

![Fig.8 Magnetic Debye material model: the two springs are connected in series](image)

We have already discussed how the elastic coefficient can be viewed as the reciprocal of the magnetic permeability. Now we introduce the time constant of the magnetic Debye material $\tau$ as corresponding to the ratio $\gamma/k_1$. The mechanical model needs at this point to be further discussed. Consider a mechanical model where a spring is connected in series as shown in Figure (9), to a dissipative device characterized by a damping coefficient $\gamma$.

![Fig.9 Spring in series with a dissipative device.](image)

When the first spring (with elastic constant $k_0$) is elongated by $x_0$, and the damped oscillator allows for an elongation $x_1$, the total elongation becomes $x_{eq}=x_0+x_1$. The restoring force is constant along the direction of motion:

$$F = -k_0 x_0 = -k_i x_i - \gamma \frac{\partial x_i}{\partial t},$$  

and can be written in a more compact way as $F = -k_{eq} x_{eq}$, if we define an "equivalent spring constant"

$$k_{eq} = \left( \frac{1}{k_0} + \frac{1}{k_i + \gamma \frac{\partial}{\partial t}} \right)^{-1} = \frac{k_i (k_i + \gamma \frac{\partial}{\partial t})}{k_0 + k_i + \gamma \frac{\partial}{\partial t}},$$

where we have treated the time differential with respect to time as an operator. Therefore the restoring force becomes:
\[ F = -k_0 \left( k_1 + \gamma \frac{\partial}{\partial t} \right) + k_1 + \gamma \frac{\partial}{\partial t} \cdot x_{eq} \]  \hspace{1cm} (27) \\

or, equivalently,

\[ \frac{\partial F}{\partial t} = -k_0 \frac{\partial x_{eq}}{\partial t} - k_1 \frac{\partial x_{eq}}{\partial t} \gamma - k_0 + k_1 F \]  \hspace{1cm} (28) \\

where the last equation is the differential equation for the restoring force. The total elongation is given by the stretch of the rubber band between two neighboring pulleys

\[ x_{eq} = a \Delta \theta = a \left( \theta_{i+1} - \theta_i \right) \]  \hspace{1cm} (29) \\

or in the continuum limit, as before, the new set of equations for the system is:

\[
\begin{align*}
\frac{\partial F}{\partial t} &= -a \gamma k_0 \frac{\partial}{\partial t} \left( \frac{\partial \theta}{\partial x} \right) - a \gamma k_1 \frac{\partial \theta}{\partial x} - k_0 + k_1 F \\
\frac{\partial \omega}{\partial t} &= \frac{a^2}{I} \frac{\partial F}{\partial x} \\
\frac{\partial \theta}{\partial t} &= \omega,
\end{align*}
\]  \hspace{1cm} (30) \\

In eqs. (30) \( F \) represents the restoring force, \( k_0 \) and \( k_1 \) the elastic constant of the springs and \( \gamma \) the damping coefficient. This extension of the FitzGerald model finds an exact analogy in the vector potential formulation of the Maxwell's equations, as shown in Appendix C, giving:

\[
\begin{align*}
\frac{\partial H}{\partial t} &= -\frac{1}{\mu_0} \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial x} \right) - \frac{1}{\mu_0 \tau} \frac{\partial A}{\partial x} - \frac{\mu_0}{\mu_0 \tau} H, \\
\frac{\partial E}{\partial t} &= -\frac{1}{\varepsilon} \frac{\partial H}{\partial x}, \\
\frac{\partial A}{\partial t} &= -E,
\end{align*}
\]  \hspace{1cm} (31) \\

By comparing eq.(30) to eq.(31) it is clear that there are several other analogies between the mechanical quantities and the electrical ones, and this are collected in Table V.
Table V  Equivalence between electrical and mechanical quantities in our model.

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-H_y$</td>
<td>$F$</td>
</tr>
<tr>
<td>$\tau/(\mu_s - \mu_\infty)$</td>
<td>$\gamma_1 a^2$</td>
</tr>
<tr>
<td>$1/\mu_\infty$</td>
<td>$k_0 a^2$</td>
</tr>
<tr>
<td>$1/(\mu_s - \mu_\infty)$</td>
<td>$k_1 a^2$</td>
</tr>
</tbody>
</table>

It is clear that this model can be extended to double Debye magnetic relaxations by simply adding a second damped oscillator connected in series to the existing ones as shown in Figure (10). The first spring (with elastic constant $k_0$) connected in series, as shown in Figure 10, with two damped oscillators characterized by elastic constants $k_1$ and $k_2$, and damping coefficients $\gamma_1$ and $\gamma_2$.

Fig.10  Series representation of two magnetic Debye terms as oscillators in series

When the three springs suffer an elongation $x_0, x_1$ and $x_2$, respectively the total elongation is $x_{eq} = x_0 + x_1 + x_2$. The "effective spring constant" becomes:

$$k_{eq} = \left( \frac{1}{k_0} + \frac{1}{k_1 + \gamma_1 \frac{\partial}{\partial t}} + \frac{1}{k_2 + \gamma_2 \frac{\partial}{\partial t}} \right)^{-1}.$$  \hspace{1cm} (32)

Using the more compact notation $\vec{k}_i = k_i + \gamma_i \frac{\partial}{\partial t}$, the restoring force becomes:

$$F = -\frac{k_0 \vec{k}_1 \vec{k}_2}{k_0 \vec{k}_1 + k_1 \vec{k}_2 + k_2 \vec{k}_2} x_{eq},$$  \hspace{1cm} (33)

or, equivalently,

$$k_0 \vec{k}_1 F + k_1 \vec{k}_2 F + k_2 \vec{k}_2 F = -k_0 \vec{k}_1 \vec{k}_2 x_{eq},$$  \hspace{1cm} (34)

which is the differential equation for the restoring force $F$. As before we take the total elongation as the stretch of the rubber band between two neighboring pulleys in the continuum limit. The new set of equations for the system is:
\[
\left\{
\begin{aligned}
\gamma_1 \frac{\partial^2 F}{\partial t^2} + (\gamma_1 + \gamma_2) (k_0 + k_i) \frac{\partial F}{\partial t} + (k_i k_j + k_j k_i + k_i k_j) F &= \\
= -a^2 (k_i k_j + k_j k_i + k_i k_j) \frac{\partial \theta}{\partial x} - a^2 (k_i k_j \gamma_1 + k_j k_i \gamma_1) \frac{\partial \theta}{\partial x} - a^2 (k_i \gamma_1 + k_j \gamma_1) \frac{\partial^2 \theta}{\partial x^2}.
\end{aligned}
\right.
\tag{35}
\]

where \( F \) represents the restoring force, \( k_i, k_j \) and \( k_2 \) the elastic constants of the springs and \( \gamma_1 \) and \( \gamma_2 \) the damping coefficients. The analogy with the vector potential formulation, see Appendix C [14], is possible in this case as well:

\[
\left\{
\begin{aligned}
\frac{\partial \omega}{\partial t} &= \frac{a^2}{l} \frac{\partial F}{\partial x} \\
\frac{\partial \theta}{\partial x} &= \omega
\end{aligned}
\right.
\]

The analogies between the mechanical system and the electrical formulation are easily drawn and are collected in Table VI. At this point the extension to multiple magnetic Debye materials folds out easily. The model is built by combining in series several damped oscillators. By induction we derive the analogies shown in Table VII.

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-H\gamma)</td>
<td>(F)</td>
</tr>
<tr>
<td>(1/\mu_\infty)</td>
<td>(k_0 a^2)</td>
</tr>
<tr>
<td>(1/\gamma_1(\mu_s - \mu_\infty))</td>
<td>(k_1 a^2)</td>
</tr>
<tr>
<td>(1/\gamma_2(\mu_s - \mu_\infty))</td>
<td>(k_2 a^2)</td>
</tr>
<tr>
<td>(\tau_1/\gamma_1(\mu_s - \mu_\infty))</td>
<td>(\gamma_1 a^2)</td>
</tr>
<tr>
<td>(\tau_2/\gamma_2(\mu_s - \mu_\infty))</td>
<td>(\gamma_2 a^2)</td>
</tr>
</tbody>
</table>

Table VI Equivalence between electrical and mechanical quantities in our model.
### Table VII

<table>
<thead>
<tr>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-H_y)</td>
<td>(F)</td>
</tr>
<tr>
<td>(1 / \mu)</td>
<td>(k_o)</td>
</tr>
<tr>
<td>(\tau_n / g_n (\varepsilon_s - \varepsilon_\infty))</td>
<td>(\gamma_n)</td>
</tr>
<tr>
<td>(1 / g_n (\varepsilon_s - \varepsilon_\infty))</td>
<td>(k_n)</td>
</tr>
</tbody>
</table>

The last extension of the FitzGerald mechanical model consists in the simultaneous treatment of dielectric and magnetic Debye materials. These are extremely interesting from both a theoretical and practical point of view because of the possible application in the modeling of particular absorbing materials. The mechanical model is built combining the features needed to model the electric and magnetic frequency dependent materials. Each pulley is surmounted by one (possibly many) weighted ring coupled to the bottom supporting pulley through a friction coefficient. The action and reaction of the top rings simulates the energy storage of the polarization vector, and ensures the electrical Debye character. On the other hand the frequency dependent magnetic character is ensured by the peculiar elastic properties of the connecting rubber bands, which can be schematically represented as an ideal spring connected in series with one (possibly many) dumped oscillators. By using the many analogies we have derived before, we do not need to re-derive the equation of motion for the quite complex system, but merely add to the basic equation of motion all the necessary ingredients in a heuristic fashion:

\[
\begin{align*}
\frac{\partial \omega}{\partial t} &= \frac{\partial^2 Q}{\partial t^2} \left( \omega - \omega^{(1)} \right) \\
\frac{\partial F}{\partial t} &= -a^2 k_o \frac{\partial \theta}{\partial x} - a^2 k_s \frac{\partial \theta}{\partial x} - k_o + k_s \gamma \\
\frac{\partial \omega}{\partial t} &= \frac{\partial^2 F}{\partial x^2} \left( \omega - \omega^{(1)} \right) \\
\frac{\partial \theta}{\partial t} &= \omega
\end{align*}
\]

This extension of the FitzGerald model finds again as exact analogy in the vector potential formulation of Maxwell's Equations, which can be derived by induction from the two cases treated above. The same analogies between the mechanical quantities and the electrical ones hold, as described before in detail. By proceeding along the same lines, it is possible to build a generic complex material, characterized by multiple electric and magnetic Debye poles, by simply adding the proper number of rings and dumped oscillators to the simple mechanical model.

### 6. EIGENVALUE PROBLEM

As a validation of the two dimensional FitzGerald mechanical model we present in this section a typical eigenvalue problem, and compare the numerical results obtained with this technique with the theoretical predicted values. A rectangular resonator is excited with an electric field pulse, and, after steady state is reached, by means of Discrete Fourier
Transform (DFT) [15] the resonant frequencies are extracted. These are compared with the exact theoretical values for the $m,n$ mode calculated from [16]:

$$ f_{n,m} = \frac{c}{2\pi} \left\{ \left( \frac{n\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right\}^{1/2}, $$  

(38)

where $a$ and $b$ are the linear dimensions of the resonator and $c$ is the speed of light. In our experiment we use a square resonator ($a=b$). The space was discretized by using $33 \times 33$ cells; the dimension of each cell is 0.2 mm. A Gaussian pulse in shape is applied in the geometrical center of the resonator; the width of the pulse is 40 time steps corresponding to 20 psec. The simulation is time stepped for a long enough time so that a steady state is reached, typically 8,000 steps. DFT algorithm is used at a non zero field point, away from the center and from the boundary to obtain the frequency resonances of the square two dimensional cavity. The DFT analysis of the $z$-component of the electric field was performed using 200 points to represent a span of 50 GHz centered around 25 GHz to localize all the excited modes. The resonance frequencies are shown in Fig.11.

![Graph showing resonance frequencies for square resonator excited through an electric pulse.](image)

**Fig.11** Resonance frequencies for square resonator excited through an electric pulse.

Notice that the amplitude of each peak depends on the particular choice of the observation point, care must be used to avoid null points. The frequency of the fundamental mode is determined experimentally as the position of the first maximum in Fig.11, and it coincides within an error less than 0.1% with the analytical result calculated from eq. (38). The same analysis was repeated for every other mode and the error was always of the same order [17].

7. SCATTERING PROBLEM

As a validation of the possibility of treatment of dielectric lossy materials, we present in this section numerical experiment results for the internal electric field of a uniform,
circular dielectric cylindrical scatterer. The cylinder is assumed to be infinite in the \(z\)-direction. The incident radiation is a wave TM with respect to the cylinder symmetry axis. Because there is no variation of either scatterer geometry or incident field in the \(z\)-direction, this problem may be treated as a 2-dimensional one. A two dimensional grid of 400 by 300 mesh points is used. The cylinder axis is positioned at point \((200,125)\). Second order absorbing boundary conditions are used to truncate the grid, as shown by Clayton [18]. Grid coordinates internal to the cylinder with radius 0.06m, are given by \((m - 200)^2 + (n - 125)^2 \leq 20^2\) and are related to the dielectric parameters. All the grid points outside this grid are related to the free space parameters. The plane wave source obtained as series of Gaussian pulses for the \(z\)-component of the field is activated along a line at the mesh position \(m=100\). The program is time-stepped for a long enough time so that the plane wave is scattered from the cylinder and the scattered field reaches the observation region. DFT algorithm is used to extract the information of the field distribution of the frequency component at 1.5 GHz. In the first simulation we use the following parameters: \(\varepsilon_\text{d} = 2\varepsilon_\text{e}, \Delta s=3\text{mm}, \Delta t=0.5\text{psec}\). Results are shown in Figure 12, which graphs the amplitude of the 1.5 GHz component of the field \(E_z\) measured inside the cylinder, along its diameter perpendicular to the incident wave, after 3,500 time steps.

![Fig.12](image)

**Fig.12** Comparison between computed and exact solution of the inner field for a lossless dielectric cylinder, along the cut \(i=200\).

The exact solution, is calculated using the summed series technique as in Jones [19]. The computer solution locates the positions of all the maxima and minima of the envelope of the electric field with error less than 0.3%. For the second example the cylindrical scatterer has the same parameters, except that it is lossy with a relative dielectric constant \(\varepsilon_\text{d} = 2\varepsilon_\text{e}\) and conductivity \(\sigma_\text{e}=0.0356 \text{ S/m}\). The result of this simulation is reported in Fig. 13. An analogous simulation was run with the same geometrical parameters and the same pulse shape, but for a magnetic cylinder. The assumed relative magnetic permeability of the cylinder is \(\mu_\text{d} = 2\mu_\text{e}\). The result is shown in Figure 14 and the comparison is made with Finite Difference Time Domain (FDTD) [20, 21] calculations.
for the same case. In all the considered cases computation times and memory requirement were identical using FDTD and our technique.

Fig. 13 Comparison between computed and exact solution of the inner field for a lossy dielectric cylinder, along a cut \( i=200 \).

Fig. 14 Comparison between our model and FDTD of the inner field computed scattered field for a lossless permeable cylinder, along the cut \( i=200 \).

8. ECHO EXPERIMENTS
Let us consider a plane wave incident upon the flat infinite air-medium interface [4]. This geometry also allows a simple implementation of the analytic solution. The one dimensional space consists of 1000 cells: 700 are used to model the free space (air) and the remaining 300 are used for the complex material. Each cell corresponds to a length of
0.1 mm and the time step is 0.25 psec. The incident wave is a Gaussian pulse with maximum frequency of 200 GHz and width of 20 time steps. The pulse is launched at the cell position 300 and the DFT of the incident pulse is performed at position $m=310$ for 300 time steps. This represents the spectrum of the incident wave. The simulation is time stepped for a long enough time until the pulse reaches the interface and is partially reflected. A second DFT analysis is performed on the reflected pulse, accurately windowed, at position $m=600$ for the same number of time steps. This represents the spectrum of the reflected wave. The reflection coefficient as a function of frequency is therefore calculated as the ratio of the two spectra. The calculated reflection coefficient is compared to the corresponding analytical quantity obtained in the frequency domain from the following relation:

$$|R(\omega)| = \frac{|\eta_r - \eta_\text{m}|}{|\eta_r + \eta_\text{m}|}$$  \hspace{2cm} (39)

where $\eta_0$ and $\eta_r$ are the characteristic impedance of free space and the complex medium respectively, and are given by

$$\eta_0 = \frac{\mu_0}{\varepsilon_0} \hspace{2cm} (40a)$$

$$\eta_r = \frac{\mu_r(\omega)}{\varepsilon_r(\omega)} \hspace{2cm} (40b)$$

In the first experiment we consider the water-air interface; the complex permittivity of

![Reflection coefficient for air-water interface](image)

**Fig.15** Reflection coefficient for air-water interface
water can be approximated by a single order Debye relaxation. We have used $\varepsilon'_e = 81\varepsilon_e$, $\varepsilon''_e = 1.8\varepsilon_e$, and $\tau_0=9.4\times10^{-12}$ sec [10]. In the second experiment we studied the reflection coefficient at the interface between air and a two pole electric Debye material, for which we chose the following values: $\varepsilon'_e = 100\varepsilon_e$, $\varepsilon''_e = 4\varepsilon_e$, $\tau_1=10^{-11}$ sec, $\tau_2=5.3\times10^{-11}$ sec, $g_1=0.7$, $g_2=0.3$. The poles have been chosen in such a way that the two relaxation times are well separated. The results for the reflection coefficients are plotted in Fig. 15 and Fig. 16. In the next two experiments we consider single and double magnetic Debye materials. For the first experiment we have used $\mu'_e = 81\mu_e$, $\mu''_e = 1.8\mu_e$, and $\tau_0=9.4\times10^{-12}$ sec; while for the second experiment we have used $\mu'_m = 100\mu_m$, $\mu''_m = 4\mu_m$, $\tau_1=10^{-11}$ sec, $\tau_2=5.3\times10^{-11}$ sec, $g_1=0.7$, $g_2=0.3$. Results are shown in Figures 17 and 18, which compare the reflection coefficient calculated from the simulations to the analytical results. In the last experiment we consider complex medium characterized by an electric and magnetic Debye relaxation. The constitutive parameters of the medium are matched in order to obtain a perfect absorbing material at all frequencies (Heavyside condition).

In particular we have used $\varepsilon'_e = 81\varepsilon_e$, $\varepsilon''_e = 1.8\varepsilon_e$, and $\tau_e=9.4\times10^{-12}$ sec, $\mu'_e = 81\mu_e$, $\mu''_e = 1.8\mu_e$, and $\tau_m=9.4\times10^{-12}$ sec; such that the ratio $(\mu(\omega)/\varepsilon(\omega))^{1/2}$ is frequency independent and equal to $\eta_0$. The results reported in Figure 19 confirm the prediction of no reflection as all frequencies, even though the calculation is affected by an error below 38 dB due to the finite cell size. In fact, higher precision is achieved by decreasing the cell size up to the stability limit.

**Fig. 16** Reflection coefficient for two-pole Debye material
Fig. 17 Reflection coefficient for single magnetic Debye materials

Fig. 18 Reflection coefficient for double magnetic Debye materials
9. COMPARISON WITH OTHER NUMERICAL TECHNIQUES

The key feature of the techniques presented in this work, besides the pedagogical value
of the mechanical analogies, resides on the condensed node representation of the field components. Once space and time are discretized, in order to solve Maxwell's Equations in the vector potential (VP) formulation, all the components of the quantities involved in the model refer to the same location of the computational grid. This property is common also to other established numerical techniques such as condensed Transmission Line Matrix (TLM) [22, 23], while differs substantially from the well known Yee representation scheme exploited by FDTD[15]. In Fig. (20) we plot the field components used in the two dimensional FDTD cell (Ez, Hx, Hy) and the analogous components used in our vector potential formulation (Ez, Az).

![Fig.19](image1.png)

**Fig.19** Reflection coefficient for wide band absorbing material.

![Fig.20](image2.png)

**Fig.20** Comparison of the field components location in the elementary cell between 2-D FDTD and D-FTD.

Note that the FDTD components are dislocated along the sides of the cell, while the vector potential components are both located in the center of the cell. As a consequence of the fact that the field is condensed, our technique offers superior performance in terms of grid dispersion with respect to the FDTD formulation, as it was already proven in the
literature for the analogous TLM. At the same time no price has to be paid in terms of memory requirements for all the cases considered, and this is an advantage with respect to TLM. Furthermore for the two dimensional case considered below only two quantities need to be considered and stored in time (Ez, Az) as opposed to the FDTD representation where three components need to be used (Ez, Hx, Hy). This results in one third memory saving over FDTD, with no penalty in execution time and algorithm complexity. In the following numerical experiment we evaluate the cutoff frequency due to grid dispersion in a computer experiment and we compare it with the same quantity evaluated in the analogous FDTD experiment. A locally plane wave tilted 45° with respect to the grid main axes is generated as shown in Fig (21).

Fig.21 Field distribution and experimental geometry for the grid dispersion numerical experiment.

The wave is generated by a Gaussian pulse 62.5 fsec in width corresponding to a maximum frequency content of 200 GHz. The wave front extends for 100 cells across corresponding to 2.0 cm, and the observation point is placed along the perpendicular to the wave front, 7.07 mm away from the wave source location. This geometry was chosen so that the wave is almost plane when it leaves the observation point. The mesh size is coarse enough such that grid dispersion will occur. As shown in [24], for a wave propagating at 45° along the FDTD grid, the grid dispersion causes the wave propagation velocity to fall to zero when \( \Delta s > 0.5\lambda \), where \( \lambda \) is the wavelength of the electromagnetic wave examined. The corresponding cutoff frequency for our particular choice of \( \Delta s \) is \( f_{\text{max}}=150\text{GHz} \). The simulation is time-stepped for a long enough time so that the Gaussian pulse goes entirely past the observation point. The time responses are recorded in both cases and are shown in Fig. (22a-b).
Due to grid dispersion, the original pulse is altered and develops a ringing tail. DFT analysis gives the spectrum content which is shown in Fig. (23), where we plot the magnitude of the two spectra, and in Fig. (24a-b), where we plot the corresponding phase versus frequency. Notice the sudden drop of the magnitude at the cutoff frequency and the corresponding loss of phase linearity. The observed FDTD cutoff frequency agrees with the predicted value of 150GHz, while our technique exhibits the cutoff at 170GHz, corresponding to a 13% improvement in bandwidth. The same bandwidth (170GHz) can be achieved by FDTD if the mesh size is reduced to
\[ \Delta s = 8.5 \cdot 10^{-4} \, m \] corresponding to a memory increase of 38\% and execution time increase of 38\% \[25\].

**Fig. 24** (a) Phase response of FDTD plotted versus frequency (b) Same quantity for D-FTD. Notice that the loss in linearity of the phase with frequency marks the cutoff frequency.

10. CONCLUSIONS
We have generalized the mechanical model first proposed by FitzGerald, to account for different realistic materials. The pedagogical advantage of the mechanical analogy resides on the ability to visualize immediately the propagation mechanism of the different electromagnetic quantities and their relation to ponderable media which can be modeled simply by modifying the mechanical properties of the objects composing the medium. At all stages we have supported the several extensions of the original mechanical model with rigorous analogies with the classical vector potential formulation. In addition to this nice mechanical analogy, we note that our formulation of the problem resides essentially on the vector potential, rather than on the fields themselves. Clearly, this is not only a formal point, and the condense node representation leads to low grid dispersion and savings in term of memory and computational time requirements with respect to FDTD, for the two dimensional cases considered. Furthermore, the simplicity of the resulting equations must be stressed, because in the considered two-dimensional cases \( E \) and \( A \) exhibit only one component each at variance of \( H \). Various results have been presented in this paper to suggest a wide spectrum of possible engineering applications. We presented scattering problems from classical objects composed of dielectric, magnetically permeable, dielectrically lossy, Debye and absorbing materials, and validated our results by comparison with rigorous frequency domain canonical solutions, or FDTD calculations. The condensed node character of this time domain formulation results in lower grid dispersion with respect to FDTD. Our analysis concludes that, in order to obtain the same bandwidth, an increase in 38\% in memory size and an analogous increase
in computational time, is required for the FDTD two dimensional case. The authors extend their appreciation to Professor Engquist for his comments on the absorbing boundary condition we used.

APPENDIX A
Consider Maxwell's equations in a dielectrically lossy source free region:

\[
\begin{align*}
\nabla \times \mathbf{E} & = -\frac{\partial \mathbf{B}}{\partial t} & \text{(A1)} \\
\nabla \times \mathbf{H} & = \frac{\partial \mathbf{D}}{\partial t} + \sigma \mathbf{E} & \text{(A2)} \\
\n\nabla \cdot \mathbf{D} & = 0 & \text{(A3)} \\
\n\nabla \cdot \mathbf{B} & = 0 & \text{(A4)}
\end{align*}
\]

From \((A.4)\)

\[
\mathbf{B} = \nabla \times \mathbf{A},
\]

and combining \((A.1)\) and \((A.5)\)

\[
\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi.
\]

For a non dispersive material \(\mathbf{D} = \varepsilon \mathbf{E}\) and \((A.3)\) can be rearranged to read

\[
\nabla \cdot \mathbf{E} + \nabla \ln \varepsilon \cdot \mathbf{E} = 0
\]

which can be combined with eq. \((A.6)\) and rearranged to obtain

\[
\nabla^2 \phi - \varepsilon \mu \frac{\partial^2 \phi}{\partial t^2} + \nabla \ln \varepsilon \cdot \nabla \phi = -\frac{\partial}{\partial t} \left[ \nabla \cdot \mathbf{A} + \varepsilon \mu \frac{\partial \phi}{\partial t} + \mathbf{A} \cdot \nabla \ln \varepsilon \right].
\]

Equation \((A.8)\) couples \(\phi\) and \(\mathbf{A}\); but the two variables can be decoupled with the use of the Lorentz-Gauge condition,

\[
\nabla \cdot \mathbf{A} = -\varepsilon \mu \frac{\partial \phi}{\partial t},
\]

and for the particular geometry such that

\[
\mathbf{A} \cdot \nabla \ln \varepsilon = 0
\]

which correspond to the condition of space invariant permittivity, \(\varepsilon = \text{Const.}\) or \(\varepsilon = \varepsilon(x,y)\), and \(\mathbf{A} = \mathbf{A}_z \mathbf{z}\). If we restrict our analysis to the above case, taking the curl of both sides of eq. \((A.5)\) and combining with eq. \((A.2)\), we obtain

\[
\mu \frac{\partial (\varepsilon \mathbf{E})}{\partial t} + \sigma \mathbf{E} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} - \nabla \mu \times \mathbf{H}
\]

Therefore summarizing together eqs. \((A.6)\), \((A.9)\) and \((A.11)\) we obtain
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial (\varepsilon \mathbf{E})}{\partial t} = \nabla \times \frac{1}{\mu} \nabla \times \mathbf{A} - \sigma \mathbf{E} \\
\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} - \nabla \phi \\
\nabla \cdot \mathbf{A} = -\varepsilon \mu \frac{\partial \phi}{\partial t}
\end{array} \right. \\
\text{.} \\
\text{(A.12)}
\end{aligned}
\]

For the two dimensional case, all the space derivative in the z-direction disappear, so we obtain
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{\partial \mathbf{E}_z}{\partial t} = -\frac{1}{\varepsilon} \left[ \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial \mathbf{A}_y}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu} \frac{\partial \mathbf{A}_y}{\partial y} \right) \right] - \frac{\sigma}{\varepsilon} \mathbf{E}_z \\
\frac{\partial \mathbf{A}_y}{\partial t} = -\mathbf{E}_z \\
\text{.} \\
\text{(A.13)}
\end{array} \right.
\]

APPENDIX B

In the case of an electric Debye material, we can define the polarization in the time domain, in the following way
\[
\varepsilon \mathbf{E}_z = \varepsilon_r \mathbf{E}_z + \mathbf{P}_z \\
\text{.} \\
\text{(B.1)}
\]
where the polarization satisfies the following differential equation:
\[
\mathbf{P}_z + \tau \frac{\partial \mathbf{P}_z}{\partial t} = \varepsilon_r \mathbf{E}_z \\
\text{.} \\
\text{(B.2)}
\]
Substituting eq. (B.1) and (B.3) into (A.13), and restricting our attention to the lossless case, we obtain
\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{1}{\varepsilon_r} \frac{\partial \mathbf{P}_z}{\partial t} = -\frac{1}{\tau} \left( \frac{\mathbf{P}_z}{\varepsilon_r} - \mathbf{E}_z \right) \\
\frac{\partial \mathbf{E}_z}{\partial t} = -\frac{1}{\varepsilon_r} \left[ \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial \mathbf{A}_y}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu} \frac{\partial \mathbf{A}_y}{\partial y} \right) \right] - \frac{1}{\varepsilon_r} \frac{\partial \mathbf{P}_z}{\partial t} \\
\frac{\partial \mathbf{A}_y}{\partial t} = -\mathbf{E}_z \\
\text{.} \\
\text{(B.3)}
\end{array} \right.
\]

At this stage the derivation can be easily extended to double or multiple electric Debye relaxations. We outline here the case of double Debye, but the extension to multiple is immediate. We use the same definition of the polarization as in eq. (B.1), but now we imagine \( \mathbf{P}_z \) as the sum of two polarizations \( \mathbf{P}^{(1)}_z \) and \( \mathbf{P}^{(2)}_z \), so that the "effective" average polarization \( \mathbf{P}_z = \mathbf{P}^{(1)}_z + \mathbf{P}^{(2)}_z \). This notation comes particularly useful because of the particular form of the dielectric constant in the time domain in the case of two Debye relaxations. In fact the electric displacement can be written as:
\[ \varepsilon E_z = \varepsilon_r E_z + \varepsilon_r \left[ \frac{g_1}{1 + \tau_1 \frac{\partial}{\partial t}} + \frac{g_2}{1 + \tau_2 \frac{\partial}{\partial t}} \right] E_z, \quad \text{(B.4)} \]

where \( g_1 + g_2 = 1 \); or using the polarization

\[ P_z = \frac{\varepsilon_r g_1}{1 + \tau_1 \frac{\partial}{\partial t}} E_z + \frac{\varepsilon_r g_2}{1 + \tau_2 \frac{\partial}{\partial t}} E_z = P_z^{(1)} + P_z^{(2)}, \quad \text{(B.5)} \]

where it is clear that each polarization can be though as to act independently from the other, and obeys the differential equation

\[ P_z^{(i)} + \tau_i \frac{\partial P_z^{(i)}}{\partial t} = \varepsilon_r g_i E_z. \quad \text{(B.6)} \]

Substituting eq. (B.1) and (B.6) into (A.13), we obtain, for the lossless case,

\[
\begin{bmatrix}
\frac{1}{\varepsilon_r} \frac{\partial}{\partial t}
\end{bmatrix}
= -\frac{g_1}{\tau_1} \left( \frac{P_z^{(1)}}{\varepsilon_r g_1} - E_z \right) - \frac{g_2}{\tau_2} \left( \frac{P_z^{(2)}}{\varepsilon_r g_2} - E_z \right)
\]

\[
\begin{bmatrix}
\frac{\partial E_z}{\partial t} \\
\frac{\partial A_y}{\partial t} \\
\frac{\partial A_y}{\partial x}
\end{bmatrix}
= -\varepsilon_r \left[ \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial A_y}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{\mu} \frac{\partial A_y}{\partial y} \right) \right] - \frac{1}{\varepsilon_r} \frac{\partial P_z}{\partial t}
\]

\[
\begin{bmatrix}
\frac{\partial A_y}{\partial x}
\end{bmatrix}
= -E_z. \quad \text{(B.7)}
\]

APPENDIX C

In the case of a magnetic Debye material, we will restrict our derivation here to the one dimensional case; the extension to the two dimensional case is straight forward. In the time domain a Debye permeability can be written as follows

\[ \mu H_z = \mu_r H_z + \frac{\mu_r - \mu_m}{1 + \tau \frac{\partial}{\partial t}} H_z = \mu_r H_z + \frac{\mu_m}{1 + \tau \frac{\partial}{\partial t}} H_z. \quad \text{(C.1)} \]

Upon introduction of the vector potential

\[ B = -\frac{\partial A}{\partial x}, \quad \text{(C.2)} \]

eq. (C.1) and (C.2) are combined, to yield a differential equation for \( H_y \):

\[ \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_r} \frac{\partial A_y}{\partial x} - \frac{\partial A_y}{\partial x} - \frac{1}{\mu_m} \frac{\partial A_y}{\partial x} - \frac{\mu_m}{\mu_r} H_y. \quad \text{(C.3)} \]

In one dimension \( \frac{1}{\mu} \frac{\partial A_y}{\partial x} = -H_y \), therefore the time evolution equation for the Electric field

\[ \frac{\partial A_y}{\partial t} = -H_y. \]
\[ \frac{\partial E}{\partial t} = -\frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial A}{\partial x} \right) \]  

(C.4)

can be replaced by the equations

\[
\begin{align*}
\frac{1}{\mu} \frac{\partial A}{\partial x} &= -H, \\
\frac{\partial E}{\partial t} &= \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( H \right) 
\end{align*}
\]  

(C.5)

where it is clear that the top equation is a differential equation for \( H_y \). Combining eq. (C.5) with eq. (C.3) we obtain:

\[
\begin{align*}
\frac{\partial H}{\partial t} &= -\frac{1}{\mu} \frac{\partial A}{\partial x} - \frac{1}{\mu} \frac{\partial A}{\partial \tau} - \frac{\mu}{\mu} H, \\
\frac{\partial E}{\partial t} &= \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( H \right), \\
\frac{\partial A}{\partial t} &= -E,
\end{align*}
\]  

(C.6)

We outline here the derivation of the vector potential formulation in the case of a double magnetic Debye material. Extension to multiple Debye relaxations is straightforward. The Debye permeability assumes the form

\[ \mu H_y = \mu_0 H_y + \mu_\omega \left( \frac{g_1}{1 + \tau_1 \partial / \partial t} + \frac{g_2}{1 + \tau_2 \partial / \partial t} \right) H, \]  

(C.7)

where \( g_1 + g_2 = 1 \); with the introduction of the vector potential, this equation becomes a differential equation for \( H_y \). Following the same derivation we outlined before, we arrive at the equations for double Debye materials:

\[
\begin{align*}
\left( \mu_\omega \tau \right) \frac{\partial H}{\partial t} &= -\left( \mu_\omega + \mu_\omega \right) \tau H_y + \left( \mu_\omega + \mu_\omega \right) \tau \frac{\partial H}{\partial t} + \left( \mu_\omega + \mu_\omega \right) H_y = \\
\frac{\partial A}{\partial x} &= \left( \tau_1 + \tau_2 \right) \frac{\partial A}{\partial t} + \left( \tau_1 + \tau_2 \right) \frac{\partial A}{\partial t} \frac{\partial}{\partial x} \\
\frac{\partial E}{\partial t} &= \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( H \right), \\
\frac{\partial A}{\partial t} &= -E,
\end{align*}
\]  

(C.8)

where \( \mu_i = \mu_\omega g_i \),
REFERENCES


