Combining Beamforming and Space-Time Coding Using Quantized Feedback

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Abstract

We combine space-time coding and transmit beamforming over multiple-antenna quasi-static fading channels using resolution-constrained channel state information at the transmitter (CSIT). The combining is performed using a class of constellation sets inspired from orthogonal designs and directional beamforming or in general, unitary precoding with quantized feedback. This constellation class is called partly orthogonal designs (PODs). The conventional way of using quantized feedback at the transmitter is through precoded space-time block codes (PSTBCs) that provide full-diversity order with all system configurations in terms of the number of feedback regions and the number of transmit antennas. PODs maintain the same advantage. However, they require less decoding and quantization complexity compared to PSTBCs. We develop PODs for transmission over multiple-antenna channels, analyze their performance, and optimize their structures based on pairwise error probability analysis. PODs create a larger combining space compared to PSTBCs. We can show that with certain system configurations, PODs outperform PSTBCs. Using PODs, we also propose a combined coding, beamforming, and spatial multiplexing scheme over multiple-antenna multi-user channels that enables a low-complexity joint interference cancellation method.

Index terms

Multiple-antenna channels, space-time coding, beamforming, precoding, quantized feedback.

I. INTRODUCTION

For multiple-antenna transmission over quasi static fading channels, we assume that the receiver is able to estimate the channel coefficients perfectly. Furthermore, there is a feedback link carrying quantized channel state information (CSI) from the receiver to the transmitter. With available partial CSI at the transmitter (CSIT), directional beamforming (or precoding) can increase the average

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received signal-to-noise ratio (SNR). This benefit of beamforming is called array gain. It improves the capacity of communication systems [1], [2], reduces the outage probability [3], and enhances the error performance. Partial CSIT is widely used in the literature to adjust the shape of the transmission matrix constellations in response to the channel, e.g. using precoded space-time block codes (PSTBCs) [4]–[8], antenna subset selection [9], and the combination of beamforming and power control [10], [11].

In quasi-static environments, the channel remains constant in each data frame and changes independently afterwards. Therefore, the transmitter must acquire the CSI using a feedback link and update it in every frame. The main obstacle for this type of closed-loop communications is the bandwidth limitation of the feedback link, and the best treatment is to use quantization to reduce the feedback rate [12]–[14]. Many researchers have proven that if the feedback is error-free and delay-free, by combining space-time coding (STC) and beamforming (or precoding), obtaining the benefits of diversity order and array gain is possible with any feedback rate [9], [15]–[21]. To the best of our knowledge, however, the best performance results with rate-limited feedback are achieved by PSTBCs in [16]. These PSTBCs manipulate orthogonal space-time block codes (OSTBCs) in [22], [23] interactively and make the transmit signals coherent to the channel direction, introduced by the feedback link.

We propose a new coded modulation class called Partly Orthogonal Designs (PODs). PODs have a space-time coding part which imposes low decoding complexity at the receiver due to its orthogonal structure. Also they have a beamforming or precoding part that uses quantized channel feedback. The goal of the CSI quantization scheme used in the receiver/transmitter of a POD system is to maximize the received SNR (array gain). For this purpose, Grassmannian beamforming and unitary precoding codebook structures in the literature are two efficient and practical schemes [13], [16] [24]. Designing PODs is also inspired from these beamforming strategies [22], [23]. Nonetheless, PODs are more general combining structures compared to PSTBCs and in fact PSTBCs can be viewed as a sub-class of PODs. In contrast to the previously known PSTBCs, PODs perform the beamforming operations inside the code matrices. In many cases, the members of the POD matrix constellation class require less decoding complexity and much less quantization complexity compared to the conventional PSTBCs. Some PODs also outperform PSTBCs as we will discuss in this paper.

We also extend POD structures for transmission over multiple-antenna multi-user channels. A
multiple-access scenario is considered in this paper, where some users employ beamforming and
some users work in the open-loop mode. In this case, from the receiver’s point of view, the
aggregation of the transmitted signals from different users makes a “virtual” POD constellation
matrix. In this case, a POD can be considered as a way to combine spatial multiplexing, space-time
coding, and transmit beamforming.

The rest of this paper is organized as follows: In Section II, we introduce the components of the
system, including PODs with different configurations, feedback channel and quantizer specifications,
and decoder structures. In Section III, we analyze the performance of PODs and optimize these
code structures based on a pairwise error probability metric. Extending the scope of PODs to the
multi-user channels is presented in Section IV. In Section V, several simulation results are provided
to show the properties of PODs in different configurations. Finally, we draw our conclusions in
Section VI.

Notations: In the sequel, \( \|x\|_F \) denotes the Frobenius norm of the matrix or vector \( x \). The operators
\((\cdot)^T\), \((\cdot)^\dagger\) and \((\cdot)^*\) stand for transpose, Hermitian, and conjugate, respectively. The inner product of
two vectors of the same size is shown by \( \langle X, Y \rangle \). Finally, \( p(x) \) and \( \mathbb{E}_x \{ f(x) \} \) show the probability
density function (pdf) of a random variable and the expectation of a function with respect to that
variable, respectively.

II. SYSTEM COMPONENTS

As shown in Fig. 1 schematically, our transmission scheme consists of a multiple-antenna point-
to-point wireless forward link with \( M_t \) transmit antennas and \( M_r \) receive antennas. The input bit
stream is mapped to the baseband \( M_t \times T \) modulation symbol matrices, where \( T \) denotes the
duration of each constellation matrix in time. We normalize the power of each transmission matrix
to \( M_t T \) and we assume that the transmission power is a constant.

The channel remains constant over the duration of multiple matrix symbols. At the receiver, ideal
channel estimation is assumed. The receiver also uses vector quantization to generate a feedback
index, representing the state of the channel in each quasi-static fading block. The feedback channel is
assumed to be error-free and delay-free. Therefore, the transmitter chooses its symbol constellation
(a set of matrices to be used for modulation) based on the feedback index generated at the receiver
and combines coding and beamforming. Upon receiving the signal, knowing the feedback symbol
and thus the transmission symbol constellation, the receiver performs maximum likelihood (ML)
decoding.
A. Partly Orthogonal Designs (PODs)

In the most generic form, we build PODs on the rows of an orthogonal space-time block code (OSTBC) [22]. Suppose that the transmitter employs $M_t$ transmit antennas and the goal is to combine a $\lambda$-dimensional directional (vector) beamformer $\omega_i$ with an OSTBC. To construct a POD, we proceed the following steps:

- Pick an OSTBC matrix constellation of size $M \times T$, where $M \geq M_t - \lambda + 1$.
- Keep the first $M_t - \lambda$ rows of the OSTBC at their places. These rows construct the non-beamformed part of the POD.
- Multiply each element of the row $M_t - \lambda + 1$ by the $\lambda$-dimensional beamforming vector $\omega_i$, where $\|\omega_i\|_F^2 = 1$ and place the resulting column vectors at the bottom of the code matrix. The resulting sub-matrix constructs the beamformed part of the POD.
- Multiply the non-beamformed part of the code by the real power loading factor $\nu$ and the beamformed part by the real power loading factor $\mu$, where $\mu^2 + (M_t - \lambda)\nu^2 = M_t$ is the instantaneous power constraint at the transmitter.

For example, we can use the following orthogonal design matrix from a real constellation, $\{z_i\}$ [23]:

$$Z = \begin{bmatrix}
z_1 & z_2 & z_3 & z_4 \\
z_2 & -z_1 & z_4 & -z_3 \\
z_3 & -z_4 & -z_1 & z_2 \\
z_4 & z_3 & -z_2 & -z_1 \\
\end{bmatrix} \tag{1}$$

Using a 2-dimensional beamformer ($\lambda = 2$), the following structure is a POD:

$$Z_2 = \begin{bmatrix}
\nu z_1 & \nu z_2 & \nu z_3 & \nu z_4 \\
\nu z_2 & -\nu z_1 & \nu z_4 & -\nu z_3 \\
\mu z_3 \omega_i & -\mu z_4 \omega_i & -\mu z_1 \omega_i & \mu z_2 \omega_i \\
\end{bmatrix} \tag{2}$$

where $w_i = [w_i[1] \ w_i[2]]^T$ is the beamforming vector chosen by the feedback index. Following the same procedure, we can perform 3 and 4-dimensional beamforming for PODs as follows:

$$Z_3 = \begin{bmatrix}
\nu z_1 & \nu z_2 & \nu z_3 & \nu z_4 \\
\mu z_2 \omega_i & -\mu z_1 \omega_i & \mu z_4 \omega_i & -\mu z_3 \omega_i \\
\end{bmatrix} \tag{3}$$

$$Z_4 = \begin{bmatrix}
\mu z_1 \omega_i & \mu z_2 \omega_i & \mu z_3 \omega_i & \mu z_4 \omega_i \\
\end{bmatrix} \tag{4}$$
Later on we will show that the above matrix constellations enable separate ML decoding of symbols. After defining the POD design criteria, we will also clarify how to choose $\lambda$, $\mu$, and $\nu$. Note that the code structure in (4) is a PSTBC which employs vector codebooks, i.e., where the inner code is $1 \times T$ and the precoder matrix (vector) is $M_t \times 1$.

One can build a POD starting from bottom to top. It is sufficient to find $M_t - \lambda + 1$ orthogonal rows of a STBC matrix, keep $M_t - \lambda$ rows at their places, and multiply each element of the last row by $\omega_i$. With some $M_t$ and $\lambda$ dimensions, we cannot find $M_t - \lambda + 1$ complex orthogonal rows. In these cases one can also extend the idea of PODs to partly quasi-orthogonal designs (PQODs) using a quasi-orthogonal STBC (QOSTBC) structure [25] [26]. Remember that QOSTBCs are pairwise decodable and so are their counterparts in the PQOD class.

### B. Feedback Channel and Quantizer

The wireless channel model that we consider in this work is quasi-static, zero-mean, circular symmetric, and complex Gaussian with independent and identically distributed (i.i.d.) coefficients. We represent this multiple-antenna channel by an $M_t \times M_r$ matrix $\hat{h}$. Suppose that the $\lambda \times M_r$ channel sub-matrix $\bar{h}$ from channel matrix $\hat{h}$ is processed at the receiver and the index of a vector quantizer (VQ) encoder is conveyed back to the transmitter. The sub-matrix $\bar{h}$ is composed of the last $\lambda$ rows of $\hat{h}$, i.e., $\bar{h} = \left[ \hat{h}^T_{(M_t-\lambda) \times M_r}, \hat{h}^T_{\lambda \times M_t} \right]^T$. Here we quantize $\lambda$ rows of $\hat{h}$ in the matrix $\bar{h}$ where $\lambda \leq M_t$.

In our system, the transmitter uses constant power. In this case, the knowledge of the amplitude of $\bar{h}$, $\|\bar{h}\|_F$ cannot help the transmitter to improve the performance of the system. We instead use the direction of $\bar{h}$, denoted by $\bar{h} = \bar{h}/\|\bar{h}\|_F$. Note that the amplitude and the direction of $\hat{h}$ are independent and our quantization target is to encode $\bar{h}$ as a source matrix. Note also that the order of rows in POD code matrices is not important. Therefore, we can arbitrarily change the indices of the channel rows that are subjected to quantization.

To encode $\bar{h}$, a $\lambda$-dimensional quantization vector codebook $\mathcal{W} = \{\omega_1, \cdots, \omega_N\}$ with cardinality $N$ is defined at the transmitter/receiver ends. To select from $\mathcal{W}$, the feedback link carries $r = \log_2(N)$ information bits. The transmitter adjusts the transmission constellation matrix based on this feedback index.

We use a conventional quantization metric aiming to maximize the average received SNR, by choosing $\omega_i \in \mathcal{W}$, and finding the pair of Voronoi regions $\mathcal{V} = \{\mathcal{V}_1, \cdots, \mathcal{V}_N\}$ and reconstruction
codebook $\mathcal{W} = \{\omega_1, \cdots, \omega_N\}$, such that [13] [14]

\[
(\mathcal{V}, \mathcal{W}) = \arg \max_{i \in \{1, \cdots, N\}} \mathbb{E}_{\bar{h} \in \mathcal{V}_i} \left\{ \left\| \omega_i^\dagger \bar{h} \right\|_F^2 \right\}
\]

(5)

Each Voronoi region is delineated by

\[
\mathcal{V}_i = \left\{ \bar{h} \in \mathcal{C}^{\lambda \times M_r} ; \text{ s.t.,} \quad \left\| \omega_i^\dagger \bar{h} \right\|_F^2 \geq \left\| \omega_j^\dagger \bar{h} \right\|_F^2 \quad \forall i, j \in \{1, \cdots, N\} \right\}
\]

(6)

where $\mathcal{C}^{\lambda \times M_r}$ is the field of unit-norm $\lambda \times M_r$ complex matrices. Moreover, the centroids (members of the reconstruction codebook) can be calculated as $\omega_j = \arg \max_{\omega \in \mathcal{C}^{\lambda \times M_r}} \mathbb{E}_{\bar{h} \in \mathcal{V}_i} \left\{ \left\| \omega^\dagger \bar{h} \right\|_F^2 \right\}$. In an i.i.d. channel environment, the optimal quantization codebook as the solution to the latter maximization problem can be obtained by Grassmannian line packing [13], also expressed as

\[
\mathcal{W} = \arg \min_{\mathcal{C}^{\lambda \times M_r}} \max_{i, j \in \{1, \cdots, N\}} \left\| \omega_i^\dagger \omega_j \right\|
\]

(7)

In a special case, where $N \leq \lambda$, with a slight abuse of the notation, a simple codebook that satisfies (7) is the following:

\[
\mathcal{W} = \left\{ \omega_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T, \omega_2 = \begin{bmatrix} 0 & 1 & \cdots & 0 \end{bmatrix}^T, \cdots, \omega_N = \begin{bmatrix} 0 & \cdots & 1 & 0 \end{bmatrix}^T \right\}
\]

(8)

Using the above codebook for directional beamforming is equivalent to antenna selection among $N$ antennas. When $N > \lambda$, however, the codebook needs to be designed explicitly. In this case, we can maximize the average received SNR using (5). Alternatively, we can design the quantizer codebook based on solving (7). A simple solution to this problem is through the unitary constellation design technique in [16] (in the special case $M_t = \lambda$ and $M = 1$, where $M$ is the number of columns of the precoder matrix or the row dimension of the inner STC in a PSTBC).

After designing the quantizer, the quantizer encoder at the receiver must solve the maximization $i = \arg \max_i \left\| \omega_i^\dagger \bar{h} \right\|_F^2$ and the decoding at the transmitter is a table look-up operation based on the feedback index $i$. The quantization computational complexity at the receiver scales linearly with the source row dimension, $\lambda$.

C. Decoding of PODs

In this section, we use an example to show the decoding complexity of PODs with vector codebooks. In the code structures (2)-(4), the rows are orthogonal to each other, meaning that their inner product is zero. This property translates to separate ML decoding of symbols. For example,
suppose that $z_2$ from (2) is transmitted over the fading channel $\hat{h}$. The received vector is

$$\hat{r} = \begin{bmatrix} \nu z_1 & \nu z_2 & \nu z_3 & \nu z_4 \\ \nu z_2 & -\nu z_1 & \nu z_4 & -\nu z_3 \\ \mu z_3 \omega_i & -\mu z_4 \omega_i & -\mu z_1 \omega_i & \mu z_2 \omega_i \end{bmatrix}^\dagger \hat{h} + \pi$$

where in the MISO case $\hat{r} = [r_1 \ r_2 \ r_3 \ r_4]^T$, $\hat{h} = [h_1 \ h_2 \ h_3 \ h_4]^T$, and $\pi = [n_1 \ n_2 \ n_3 \ n_4]^T$. For decoding symbol $z_1$, it is sufficient that the decoder constructs

$$r_1 h_1^* - r_2 h_2^* - r_3 \left[ h_3 \ h_4 \right] \omega_i^* = z_1 \left( \nu^2 |h_1|^2 + \nu^2 |h_2|^2 + \mu^2 \left| \left< \omega_i, \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} \right> \right|^2 \right) + n_1'$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product and $n_1'$ denotes the equivalent noise variable. Note that the decoding of $z_1$ is independent from that of $z_2$, $z_3$, and $z_4$. Similar arguments can be applied to the decoding of $z_2$. To decode $z_3$, the decoder needs to construct

$$r_1 \left[ h_3^* \ h_4^* \right] \omega_i + r_3^* h_1 - r_4^* h_2 = z_3 \left( \nu^2 |h_1|^2 + \nu^2 |h_2|^2 + \mu^2 \left| \left< \omega_i, \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} \right> \right|^2 \right) + n_3'$$

Therefore, $z_3$ is decoded independently from $z_1$, $z_2$, and $z_4$. The decoding of $z_4$ is similar to $z_3$.

The only added operation compared to the decoding of open-loop OSTBCs is to calculate the product $\left[ h_3 \ h_4 \right] \omega_i^*$. This operation requires at most $\lambda$ complex multiplications, when we use one receive antenna. If $\omega_i$ contains one element equal to 1 and all others equal to 0, like in (8), this operation does not require any complex multiplications.

D. PODs with Matrix Codebooks

The original PSTBCs in [16] use $M \times T$ OSTBC constellation matrices as inner codes and multiply them by an $M_t \times M$ precoder matrix $P_i$, which is a member of the precoding matrix codebook $\mathcal{P} = \{P_1, \cdots, P_N\}$, with cardinality $N$. The objective of the precoder codebook design is again to maximize the average received SNR. As mentioned before, the practical codebook design strategy of [16] borrows constellation matrices from [27]. For arbitrary precoder dimensions in general, this technique results in a specific precoder structure that performs well with any feedback rate. The complexity of codebook design exponentially increases with the number of transmit antennas and the number of feedback regions. Love, et al. proved that if $N \geq \frac{M_t}{M}$ and the columns of the precoding codebook $\mathcal{P}$ (when all the $M_t \times M$ members are stacked in a matrix) span $\mathbb{C}^{M_t}$, then the PSTBC provides full-diversity order.
Inspired from PSTBCs, we construct a POD using matrix precoders instead of vector beamformers. The new construction algorithm is similar to the previous one. Suppose that the goal is to combine a $\lambda \times M$-dimensional matrix precoder $P_i$ with an OSTBC, when $\lambda \leq M_t$. We proceed the following steps to design a POD:

- Pick an OSTBC matrix constellation of size $(M_t - \lambda + M) \times T$.
- Keep the first $M_t - \lambda$ rows of the OSTBC at their places. These rows construct the non-precoded part of the POD.
- Multiply the remaining sub-matrix of $M$ rows by the $\lambda \times M$ precoder matrix $P_i$ and place the result at the bottom of the code matrix. The resulting rows construct the precoded part of the POD.
- Multiply the non-precoded part by the real power loading factor $\nu$ and the precoded part by the real number $\mu$.

The above structure degenerates to a PSTBC when $M_t = \lambda$. Overall, in PODs coding and beamforming or precoding are combined. However, there is no separate coding part in PSTBCs. In this sense, PODs provide more solutions for combining coding and beamforming compared to PSTBCs. We can find some cases where PODs outperform PSTBCs, as will be shown in Section V.

Decoding PSTBCs is more complex than that of open-loop OSTBCs. The added complexity is to calculate the equivalent channel matrix by constructing $P_i^\dagger \hat{h}$ at the receiver. The number of complex multiplications to calculate the equivalent channel scales with the precoder dimension. For example, with $M_t \times M$ precoders, this operation requires $M_t \times M$ complex multiplications. For PODs with matrix codebooks, the same equivalent channel calculation is required with $\lambda \times M$ precoder matrices. Therefore, the number of additional operations scales with $\lambda \times M$. Note that compared to the decoding of PODs with vector codebooks, we need more operations since in the case of PODs with vector codebooks, the added complexity is at most $\lambda$ complex multiplications. Note also that in MIMO applications, the number of additional complex operations is counted per receive antenna.

III. PERFORMANCE ANALYSIS

Using pairwise error probability analysis, we analytically optimize POD structures for transmission over MISO channels. We also analytically extend the proof of full-diversity order for transmission over MIMO channels with matrix or vector codebooks. Then we numerically optimize the generally extended PODs with different antenna configurations using Monte Carlo simulations.
A. Analysis and Optimization of PODs with Vector Codebooks Over MISO Channels

Upon transmitting the POD matrix \( \mathbf{Z} \), the \( T \times 1 \) received signal can be modelled \( \mathbf{r} = \mathbf{Z}^\dagger \hat{\mathbf{h}} + \mathbf{n} \), where \( \mathbf{n} \) is the complex and additive white Gaussian noise with variance \( \sigma_n^2/2 = M_t/(2 \text{SNR}) \) on the real and imaginary parts. Based on our normalization assumption, the SNR is defined as the ratio of the transmission power over the average noise power per (the only) receive antenna.

The conditional PEP of decoding in favor of \( \mathbf{Z}' \) when \( \mathbf{Z} \) is transmitted can be expressed as
\[
P(\mathbf{Z} \rightarrow \mathbf{Z}'|\hat{\mathbf{h}}) = Q \left( \sqrt{\frac{1}{2\sigma_n^2}} \left\| \hat{\mathbf{h}}^\dagger (\mathbf{Z} - \mathbf{Z}') \right\|_F \right),
\]
where \( Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp \left( -\frac{y^2}{2} \right) dy \) is the Gaussian tail function [23]. This expression can be tightly upper bounded in moderate to high SNR values by the Chernoff bound \( \frac{1}{2} \exp \{-d^2(\mathbf{Z}, \mathbf{Z}')/4\sigma_n^2\} \), where \( d^2(\mathbf{Z}, \mathbf{Z}') = \hat{h}^\dagger (\mathbf{Z} - \mathbf{Z}')(\mathbf{Z} - \mathbf{Z}')^\dagger \hat{h} \).

Suppose that the receiver provides the transmitter with \( \log_2(N) \) feedback bits to represent the direction of \( \hat{\mathbf{h}} \) using the vector codebook \( \mathcal{W} = \{\omega_1, \cdots, \omega_N\} \). We decompose \( \hat{\mathbf{h}} \) into its amplitude and its direction through \( \hat{\mathbf{h}} = \sqrt{\rho} \bar{\mathbf{h}} \), where \( \rho = \sum_{i=1}^{\lambda} \hat{h}[i]^2 \) is the amplitude square and \( \bar{\mathbf{h}} \) is the direction. With a zero-mean complex Gaussian channel, the vector \( \bar{\mathbf{h}} \) is uniformly distributed over the unit-norm complex sphere \( \mathcal{C}^\lambda \), [12], and \( \rho \) is a Chi-square random variable with \( 2\lambda \) degrees of freedom, characterized by \( p(\rho) = \rho^{\lambda-1} e^{-\rho}/(\lambda-1)! \).

The PEP, conditioned on the feedback information can be expressed as:
\[
P(\mathbf{Z} \rightarrow \mathbf{Z}') = \mathbb{E}_{\hat{\mathbf{h}}|\mathcal{W}} \left\{ \frac{1}{2} \exp \left( -\frac{1}{4\sigma_n^2} \left[ \hat{\mathbf{h}}^\dagger \bar{\mathbf{h}} \right] (\mathbf{Z} - \mathbf{Z}')(\mathbf{Z} - \mathbf{Z}')^\dagger \left[ \hat{\mathbf{h}} \bar{\mathbf{h}} \right] \right) \right\}
\]
where the sub-script \( \hat{\mathbf{h}}|\mathcal{W} \) reflects the feedback operation. For PODs from constellation \( \{z_i\} \), by some straightforward manipulations, the above expression can be expanded as:
\[
\text{CGM} \triangleq P(\mathbf{Z} \rightarrow \mathbf{Z}') = \mathbb{E}_{\hat{\mathbf{h}}} \left\{ \frac{1}{2} \exp \left( -\Delta_z \left[ \mu^2 \rho \zeta + \nu^2 \left\| \bar{\mathbf{h}} \right\|_F^2 \right] \right) \right\}
\]
where \( \zeta = \max_{i \in \{1, \cdots, N\}} \left| \langle \bar{\mathbf{h}}, \omega_i \rangle \right|^2 \) and \( \Delta_z = \frac{\sum_{i=1}^N |z_i - z|^2}{4\sigma_n^2} \) is proportional to the inner codes Euclidian distance and the SNR. Equation (12) is the definition of coding gain metric (CGM) in our analysis and design approach. To proceed, we write the CGM in an integral form as:
\[
\text{CGM} = \frac{1}{2} \int_{\mathbb{R}^+} \int_{\Phi_{\hat{\mathbf{h}}}} \int_{\Phi_{\zeta}} \exp \left( -\Delta_z \left[ \mu^2 \rho \zeta + \nu^2 \left\| \bar{\mathbf{h}} \right\|_F^2 \right] \right) p(\rho, \bar{\mathbf{h}}, \zeta) \, d\bar{\mathbf{h}}
\]
where, \( \mathbb{R}^+ \) means non-negative real and \( \Phi_{\bar{\mathbf{h}}} \) is the domain for the un-quantized portion of the channel vector, and \( \Phi_{\zeta} \) is the domain of \( \zeta \).

For independent and identically distributed (i.i.d.) channel realizations, the unquantized part of
the channel, \(\tilde{h}\) is independent from the quantized part, \(\overline{h}\). Furthermore, the channel direction, \(\overline{h}\) is independent from its amplitude, \(\rho\). Therefore, \(p(\rho, \tilde{h}, \overline{h})d\tilde{h} = p(\overline{h}) \, p(\rho) \, d\overline{h} \, d\zeta \, d\rho\).

**Lemma 1:** Using quantized vector beamforming, when \(\lambda \leq N\), the CGM in (13) can be approximated by:

\[
\text{CGM} \approx \frac{1/2}{1 + \mu^2 \Delta z} \left( \frac{1}{1 + (\mu^2 \Delta z)(1 - \beta)} \right)^{\lambda-1} \left( \frac{1}{1 + \nu^2 \Delta z} \right)^{M_t - \lambda} \tag{14}
\]

where \(\beta = 2 \frac{-\log_2(N)}{\lambda - 1}\) is a specification of the reconstruction codebook \(\mathcal{W}\), as defined in the proof.

**Proof:** To quantify CGM in (13), first we should derive the pdf of \(\zeta = \max_{i \in \{1, \ldots, N\}} \|\overline{h}, \omega_i\|^2\). To facilitate the derivations, when \(N \geq \lambda\), we adopt a high-resolution quantizer assumption [28].

Based on this assumption for our source vector \(\overline{h}\), all the Voronoi regions can be characterized by spherical caps around the reconstruction vector end points \(\omega_i\), where the all-zero vector is the origin. The intuition behind this assumption is that with \(N \geq \lambda\) non-overlapping conical regions in a \(\lambda\)-dimensional space, we can approximately tile the whole unit-norm \(C^\lambda\) hyper-sphere. \(^1\) The spatial gaps as the result of this tiling diminish as the quantization rate increases.

The latter spherical cap regions can be parameterized by \(S_{\omega_i}(z) = \left\{ \overline{h} \in C^\lambda \ ; \ 1 - \left\| \omega_i^\dagger \overline{h} \right\|^2 \leq z \right\}\), [29], where we call \(0 < z < 1\) the **chordal radius** [13]. Furthermore, the whole \(\lambda\)-dimensional unit-norm complex sphere \(C^\lambda\) can be spanned by \(\bigcup_{i=1}^N S_{\omega_i}(z)\). Instead of \(\zeta\) that characterizes \(\mathcal{W}_i\) from (6), we consider the cumulative distribution function of \(z\) denoted by \(F_Z(z)\) that characterizes \(S_{\omega_i}(z)\). Using the surface area of the spherical cap region \(S(z)\) in [12], \(A(S(z)) = \frac{2 \pi^2 \lambda^\lambda}{(\lambda - 1)!}\), it is shown in [29] that

\[
F_Z(z) < \bar{F}_Z(z) = \sum_{i=1}^N \frac{A(S_{\omega_i}(z))}{A(S(z))} = \begin{cases} \gamma z^{\lambda-1} & 0 \leq z < \beta \\ 1 & \beta \leq z \geq 1 \end{cases} \tag{15}
\]

where \(\beta = \left(1 - \max_{i,j} \|\omega_i^\dagger \omega_j\| \right)/2\) is called the **conforming radius** and equals to the largest value of \(z\) that does not make overlapping Voronoi regions, and \(\gamma = \beta^{-(\lambda-1)}\). For large \(N\), the conforming radius is \(\beta = 2 \frac{-\log_2(N)}{\lambda - 1}\) [12] [29].

Assuming that the inequality in (15) is tight, the CGM in (12) can be expressed as:

\[
\text{CGM} \approx \frac{1}{2} \int_0^\infty \int_0^1 d\rho \int_0^1 d\bar{F}_Z(z) \int_0^{\infty} d\theta \frac{\rho^{(\lambda - 1)} e^{-\rho}}{(\lambda - 1)!} p_\theta(\theta) \exp \left(-\mu^2 \Delta z \rho(1 - z)\right) \exp \left(-\nu^2 \Delta z \theta\right) \tag{16}
\]

\(^1\)In contrast to this scenario, when \(N < \lambda\), the optimal quantization codebook is:

\[
\left\{ \omega_1 = [1 \ 0 \ \cdots \ 0 \ 0]^T , \cdots , \omega_N = [0 \ 0 \ \cdots \ 1 \ \cdots \ 0]^T \right\}
\]

With the above codebook, the tail of the quantization subject, \(\overline{h}\) does not affect the quantization metric. Therefore, non-overlapping Voronoi regions that can tile \(C^\lambda\) cannot be defined.
In the above integral, \( \vartheta = \| \mathbf{h} \|_F^2 \) follows a Chi-square distribution with \( 2(M_t - \lambda) \) degrees of freedom. Note that from [23]- (3.19) we have \( \int_0^\infty p_\vartheta (\vartheta) \exp \left( -\nu^2 \Delta_z \vartheta \right) d\vartheta = (1 + \nu^2 \Delta_z)^{-(M_t - \lambda)} \).

To simplify the presentation, let us use the following auxiliary variable:

\[
I = \frac{1}{2} \int_0^1 \frac{dF_Z(z)}{(1 + \mu^2 \Delta_z (1 - z))^{\lambda}}< \frac{1}{2} \int_0^1 \frac{dF_Z(z)}{(1 + \mu^2 \Delta_z (1 - z))^{\lambda}} \quad (17)
\]

where the inequality can be confirmed by integration by part. By a simple change of variable \( 1 + \mu^2 \Delta_z (1 - z) = u \), the left hand side of (17) can be reexpressed as:

\[
I = \frac{1}{2} \int_0^{\beta^2} \frac{\gamma (\lambda - 1)}{[1 + \mu^2 \Delta_z (1 - z)]^{\lambda}} = \frac{\gamma}{1 + \nu^2 \Delta_z} \left( 1 + \frac{\beta}{1 + (\mu^2 \Delta_z) (1 - \beta)} \right)^{\lambda-1} \quad (18)
\]

which shows that the overall CGM can be approximated by Equation (14).

We use the result of Lemma 1 to find the optimal row dimension of the beamforming part, \( \lambda \), and the optimal power loading factors \( \mu \) and \( \nu \) in POD structures. Later on, using numerical analysis, we will confirm these results through Monte Carlo simulations.

**Theorem 1:** Based on the CGM expression in (14), we can prove the following properties:

1) When \( \lambda \leq N \), PODs are full-diversity order constellations.

2) The PEP performance of PODs is improved by increasing the feedback rate.

3) In the high SNR region, the optimal power allocation strategy is to spend the same amount of power on the beamformed part and the non-beamformed parts of PODs, i.e., \( \mu = \sqrt{\lambda} \).

**Proof:** In the high SNR region, we can reformulate the CGM expression in (14) as:

\[
CGM = \frac{1}{2} (\Delta_z \mu^2)^{-\lambda} [\Delta_z (1 - \beta) \mu^2]^{-\lambda + 1} [\Delta_z \nu^2]^{-M_t + \lambda}
\]

Note that the instantaneous power constraint at the transmitter yields \( \mu^2 + (M_t - \lambda) \nu^2 = M_t \). Therefore, at high-SNR we have

\[
CGM = \frac{1}{2} \Delta_z^{-M_t} (1 - \beta)^{-\lambda + 1} (\mu^2)^{-\lambda} \left( \frac{M_t - \mu^2}{M_t - \lambda} \right)^{M_t + \lambda} \quad (19)
\]

The first property is clear from the definition of the diversity order.

\[
g_d = - \lim_{\Delta_z \to \infty} \frac{\log [P(Z \to Z' | i)]}{\log [\Delta_z]} = M_t \quad (20)
\]
The second property can be proved by noting the role of the conforming radius \( \beta \) in the CGM expression. As the number of quantization bits increases, \( \beta \) becomes smaller. Therefore CGM decreases and POD constellations provide better performance.

To prove the third property we investigate the behavior of CGM with respect to \( \mu^2 \). Minimizing CGM is a simple constraint optimization. First we calculate the first-order derivative of CGM with respect to \( \mu^2 \),

\[
\frac{d\text{CGM}}{d(\mu^2)} = \text{Const.} \left[ -\lambda (\mu^2)^{-\lambda-1} \left( \frac{M_t - \mu^2}{M_t - \lambda} \right)^{-M_t+\lambda} + (\mu^2)^{-\lambda} \left( \frac{M_t - \mu^2}{M_t - \lambda} \right)^{-M_t+\lambda-1} \right]
\]

Note that \( \mu^2 \neq 0 \). Further, we assume that \( \lambda < M_t \). Because the solution for \( \lambda = M_t \) is obviously \( \mu^2 = \lambda = M_t \) and \( \nu^2 = 0 \). Moreover, we ignore the case \( \mu^2 = M_t \) since it only associates to \( \lambda = M_t \). Setting the first-order derivative equal to zero yields

\[-\lambda (M_t - \mu^2)/(M_t - \lambda) + \mu^2 = 0 \]

The solution to the above equation and the power constraint is \( \mu^2 = \lambda \) and \( \nu^2 = 1 \).

It is straightforward to show that the second-order derivative of the CGM is always positive. So the CGM is convex with respect to \( \mu^2 \) and the latter power loading strategy is globally optimal.

The performance of PODs is improved by increasing the beamforming dimension, as long as \( \lambda \leq N \). This is shown in Fig. 2 by plotting CGM versus SNR from (14) for different parameters \( \lambda \). Fig. 2 depicts these plots for an \( M_t = 6 \) antenna system, when \( N = 4 \) feedback regions are used. From this figure we conclude that minimizing the PEP requires us to choose \( \lambda = \min\{M_t, N\} \). By this choice in a POD structure, we obtain full-diversity order and minimum pairwise error probability. Fig. 2 also shows the optimality of the third property in Theorem 1 in the finite SNR regimes. We deviated \( \mu \) slightly from \( \sqrt{\lambda} \) and observed that the CGM increases. We can see that even in low SNR regions CGM is globally minimum when \( \mu = \sqrt{\lambda} \).

B. Analysis of PODs With Matrix or Vector Codebooks Over MIMO Channels

The following theorem defines the relationship between the design parameters \( \lambda \), \( M \), and \( N \) in a full-diversity system.

**Theorem 2:** If \( N \geq \frac{\lambda}{M} \) and the columns of the precoder codebook \( \mathcal{P} \) span \( \mathcal{C}^\lambda \), then the PODs that use this codebook provide full-diversity order.

**Proof:** With a slight abuse of the notation, let \( Q_i = \omega_i \) with vector beamformers or \( Q_i = \mathcal{P}_i \) with matrix precoders. For a POD structure that uses \( \lambda \)-dimensional beamformer vectors \( \omega_i \) or \( \lambda \times M \)
precoder matrices $\mathcal{P}_i$ to quantize $\tilde{h}$, we define the coding gain metric as:

$$\text{CGM} \triangleq \mathbb{E}_{\tilde{h}|h} \left\{ \frac{1}{2} \exp \left( -\frac{1}{4\sigma_n^2} \left[ \tilde{h}^\dagger_{(M_t-\lambda)\times M_r} h^\dagger_{\lambda\times M_r} \right] (Z - Z') (Z - Z')^\dagger \left[ \frac{\tilde{h}^\dagger_{(M_t-\lambda)\times M_r}}{h^\dagger_{\lambda\times M_r}} \right] \right) \right\}$$

Using our quantization scheme, every Voronoi region can be also defined as:

$$V_i = \left\{ h; \text{s.t.,} \| Q_i^\dagger h \|_F \geq \| Q_j^\dagger h \|_F \right\} : \forall i, j \in \{1, \cdots, N\}$$

With i.i.d. channels, $\tilde{h}$ and $h$ are independent and the CGM expression can be expanded as:

$$\text{CGM} \triangleq \frac{1}{2} \mathbb{E}_{h} \left\{ \exp \left( -\mu^2 \Delta_z \left\{ \max_{i \in \{1, \cdots, N\}} \| Q_i^\dagger h \|_F^2 \right\} \right) \right\} \mathbb{E}_{\tilde{h}} \left\{ \exp \left( -\nu^2 \Delta_z \| \tilde{h} \|_F^2 \right) \right\}$$

$$= \frac{1}{2} \mathbb{E}_{h} \left\{ \exp \left( -\mu^2 \Delta_z \left\{ \max_{i \in \{1, \cdots, N\}} \| Q_i^\dagger h \|_F^2 \right\} \right) \right\} \mathbb{E}_{\tilde{h}} \left( \frac{1}{1 + \nu^2 \Delta_z} \right)^{(M_t-\lambda)\times M_r} \zeta(\Delta_z) \right) \right\} \right) \right)$$

$$= \frac{1}{2} \mathbb{E}_{h} \left\{ \exp \left( -\mu^2 \Delta_z \left\{ \max_{i \in \{1, \cdots, N\}} \| Q_i^\dagger h \|_F^2 \right\} \right) \right\} \mathbb{E}_{\tilde{h}} \left( \frac{1}{1 + \nu^2 \Delta_z} \right)^{(M_t-\lambda)\times M_r} \zeta(\Delta_z) \right) \right) \right) \right) \right)$$

Note that in PODs with vector codebooks, $M = 1$. We also mentioned that $\lambda \leq N$. Moreover, in PODs with matrix codebooks $\frac{\lambda}{M} \leq N$. Inspired from [16]-Theorem 1, we know that $-\lim_{\Delta_z \to \infty} \frac{\log[\zeta(\Delta_z)]}{\log[\Delta_z]} = \lambda M_r$. Therefore, it is easy to see that

$$g_d = -\lim_{\Delta_z \to \infty} \frac{\log[\text{CGM}]}{\log[\Delta_z]} = M_t M_r$$

and therefore, PODs provide full-diversity order. Note that using unitary constellations borrowed from [27] for precoding satisfies the assumption of this theorem.

We can also show the following properties of PODs by Monte Carlo simulations:

- Using $\lambda$-dimensional vector beamformers with $M_t - \lambda$ non-beamformed rows, the optimal power loading parameters are $\mu^2 = \lambda$ and $\nu^2 = 1$. For this structure, the best value of the beamformer dimension is $\lambda = \min\{M_t, N\}$.

- Using $\lambda \times M$ precoders with $M_t - \lambda$ non-precoded rows, the optimal power loading parameters are $\mu^2 = \frac{\lambda}{M}$ and $\nu^2 = 1$. With this structure, if $M_t > N$, the best dimension of the precoder matrix is $\lambda \times M$, with the number of inner code rows $M = \lfloor \frac{M_t}{N} \rfloor$, the largest integer smaller than $\frac{M_t}{N}$ and with the spatial dimension $\lambda = \min\{M_t, NM\}$.

C. More POD Examples

In this example, we design PODs with both vector and matrix codebooks. In Section V, we use this example to compare the performance of different POD configurations. Suppose that the
transmitter employs $M_t = 5$ antennas and the feedback codebook defines $N = 2$ quantization regions. Furthermore, suppose that we use BPSK signalling and real orthogonal design as the inner code. A conventional PSTBC uses a $3 \times 4$ orthogonal inner code and a $5 \times 3$ precoder codebook $P = \{P_1, P_2\}$:

$$Z_{\text{PSTBC}} = \sqrt{\frac{5}{3}} P_i \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_3 & -z_4 & -z_1 & z_2 \\ z_2 & -z_1 & z_4 & -z_3 \\ \end{bmatrix}$$

(23)

The scalar coefficient $\mu = \sqrt{\frac{5}{3}}$ is chosen to satisfy the power constraint, assuming that the precoder codebook is built as in [16]. Using a vector codebook $W = \{\omega_1, \omega_2\}$ of cardinality 2, we pick the beamforming dimension $\lambda = 2$ and use the following POD structure:

$$Z_{\text{POD}} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_3 & -z_4 & -z_1 & z_2 \\ z_2 & -z_1 & z_4 & -z_3 \\ \sqrt{2} z_2 \omega_i & -\sqrt{2} z_1 \omega_i & \sqrt{2} z_4 \omega_i & -\sqrt{2} z_3 \omega_i \end{bmatrix}$$

(24)

where the optimality of $\mu = \sqrt{2}$ is verified numerically. The POD structure in (24) outperforms the PSTBC in (23). Note that in this case, $\frac{M_t}{M} < N$. We can alternatively build a POD with a matrix codebook $P = \{P_1, P_2\}$, using $M \times T = 2 \times 4$ OSTBC inner codes and $\lambda \times M = 4 \times 2$ precoders and leaving $(M_t - \lambda) \times T = 1 \times 4$ dimensions without precoding:

$$Z_{\text{POD matrix}} = \begin{bmatrix} z_4 & -z_2 & -z_1 \\ z_3 & -z_2 & -z_1 \\ \sqrt{2} P_i \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & -z_1 & z_4 & -z_3 \end{bmatrix} \end{bmatrix}$$

(25)

In this case we can show numerically that $\mu = \sqrt{\frac{4}{2}}$ minimizes the error rate. In Section V, we show that the POD in (25) outperforms the other structures shown in (23)-(24).

IV. Extension to Multiple-Antenna Multi-User Channels

Based on the application of PODs in transmission over MIMO channels, we use PODs for multi-user (multiple-access) communications. We consider transmission in the uplink of a wireless network, where the base station unit decodes independent information symbols from different users using multiple receive antennas. We assume that some users in the network can employ beamforming and some users can only transmit in the open-loop mode. Normally, the major difficulty for implementing such a system is the large complexity of maximum-likelihood decoding. Inspired from the design methodologies in [30], [31], we develop a low-complexity joint interference cancellation
technique that provides the users with a diversity order that scales with the number of transmit antennas at each user and the number of receive antennas at the access point.

We describe our design technique using an example. Suppose that there are $J = 2$ users, each employing $M_t = 2$ transmit antennas. User 2 can employ beamforming, whereas User 1 can only transmit in the open-loop mode. In this example we use QPSK signalling and we assume that the number of feedback regions for User 2 is $N = 2$. Therefore, User 2 uses the vector codebook $\mathcal{W} = \{\omega_1, \omega_2\}$ for beamforming. According to the discussions in the previous sections, an SNR maximizing codebook for this case is $\mathcal{W} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T \right\}$.

In [30], the authors provided an open-loop transmission and decoding technique using Alamouti STBCs, which was simplified and used in [31] for larger systems. We extend these design techniques to our multi-user beamforming scenario using PODs. The transmit signals from both users form the following POD constellation:

$$Z_{MU} = \begin{bmatrix} -z_1(1) & z_2(1)^* \\ z_2(1) & z_1(1)^* \\ \cdots & \cdots \\ -\sqrt{2}z_1(2)\omega_i & \sqrt{2}z_2(2)^*\omega_i \end{bmatrix}$$

(26)

where the numbers in the parentheses denote the index of the users and the columns are indexed by time. User 1 transmits the upper sub-matrix and User 2 transmits the lower sub-matrix of (26). The receiver is capable of estimating the channel matrix

$$\hat{h} = \begin{bmatrix} \tilde{h} \\ \cdots \\ \tilde{h} \end{bmatrix} = \begin{bmatrix} h_{1,1}(1) & h_{1,m}(1) \\ h_{1,2}(1) & h_{1,m}(1) \\ \cdots & \cdots \\ h_{1,1}(2) & h_{1,m}(2) \\ h_{1,2}(2) & h_{1,m}(2) \end{bmatrix}$$

(27)

where the first index denotes the transmit antenna and the second one denotes the receive antenna. We have shown the second receive antenna by $m$ so that we can describe the decoding algorithm for any number of receive antennas. The code structure in (26) in fact combines beamforming, space-time coding, and spatial multiplexing. Because it transmits 4 symbols from 2 independent antenna subsets with rate 2 symbols per time slot. Note that since PSTBCs combine all the code rows together, they cannot be used to transmit independent symbols from different antennas. In other words, PSTBCs cannot be used for beamforming over multi-user channels. The constellation matrix in (26) belongs to a sub-class of PODs, where we have a smaller number of columns compared to
of users in the above sub-network, we can eliminate the interference of User m. The cancellation technique makes the decoding more practical since it provides the system with single coding complexity of ML search can become prohibitively large. Therefore, the above interference

Note that this technique incurs noise enhancement compared to ML decoding. However, the decoding complexity of ML search can become prohibitively large. Therefore, the above interference cancellation technique makes the decoding more practical since it provides the system with single decodability. As long as the number of receive antennas is greater than or equal to the number of users in the above sub-network, we can eliminate the interference of User m on the outgoing

In order to decode the transmit signals from Users 1 and 2, the receiver constructs the following equivalent channel matrices:

\[
H_1(1) = \begin{bmatrix}
-h_{1,1}(1)^* & h_{2,1}(1)^* \\
h_{2,1}(1) & h_{1,1}(1)
\end{bmatrix}, \quad H_m(1) = \begin{bmatrix}
-h_{1,m}(1)^* & h_{2,m}(1)^* \\
h_{2,m}(1) & h_{1,m}(1)
\end{bmatrix}
\]

and the following auxiliary matrices:

\[
\Omega_1(2) = \begin{bmatrix}
-\sqrt{2} \left[ \omega_i(1)h_{1,1}(2)^* + \omega_i(2)h_{2,1}(2)^* \right] & 0 \\
0 & \sqrt{2} \left[ \omega_i(1)^*h_{1,1}(2) + \omega_i(2)^*h_{2,1}(2) \right]
\end{bmatrix},
\]

\[
\Omega_m(2) = \begin{bmatrix}
-\sqrt{2} \left[ \omega_i(1)h_{1,m}(2)^* + \omega_i(2)h_{2,m}(2)^* \right] & 0 \\
0 & \sqrt{2} \left[ \omega_i(1)^*h_{1,m}(2) + \omega_i(2)^*h_{2,m}(2) \right]
\end{bmatrix}
\]

The interference of the User 1’s signals can be eliminated using

\[
\frac{H_m(1)^\dagger}{|h_{1,m}|^2 + |h_{2,m}|^2} \begin{bmatrix}
r_{1,m}^* \\
r_{2,m}
\end{bmatrix} - \frac{H_1(1)^\dagger}{|h_{1,1}|^2 + |h_{2,1}|^2} \begin{bmatrix}
r_{1,1}^* \\
r_{2,1}
\end{bmatrix} =
\]

\[
\left( \frac{H_m(1)^\dagger}{|h_{1,m}|^2 + |h_{2,m}|^2} \frac{H_1(1)^\dagger}{|h_{1,1}|^2 + |h_{2,1}|^2} \right) \begin{bmatrix}
z_1(2) \\
z_2(2)
\end{bmatrix} + \begin{bmatrix}
n_1^*(2) \\
n_2^*(2)
\end{bmatrix}
\]

Similarly, the interference of User 2 can be suppressed by

\[
\frac{\Omega_m(2)^\dagger}{2 \left| \omega_i(1)h_{1,m}(2)^* + \omega_i(2)h_{2,m}(2)^* \right|^2} \begin{bmatrix}
r_{1,m}^* \\
r_{2,m}
\end{bmatrix} - \frac{\Omega_1(2)^\dagger}{2 \left| \omega_i(1)h_{1,1}(2)^* + \omega_i(2)h_{2,1}(2)^* \right|^2} \begin{bmatrix}
r_{1,1}^* \\
r_{2,1}
\end{bmatrix} =
\]

\[
\left( \frac{\Omega_m(2)^\dagger}{2 \left| \omega_i(1)h_{1,m}(2)^* + \omega_i(2)h_{2,m}(2)^* \right|^2} \frac{H_m(1)}{2 \left| \omega_i(1)h_{1,1}(2)^* + \omega_i(2)h_{2,1}(2)^* \right|^2} \right) \begin{bmatrix}
z_1(1) \\
z_2(1)
\end{bmatrix} + \begin{bmatrix}
n_1^*(1) \\
n_2^*(1)
\end{bmatrix}
\]

Note that this technique incurs noise enhancement compared to ML decoding. However, the decoding complexity of ML search can become prohibitively large. Therefore, the above interference cancellation technique makes the decoding more practical since it provides the system with single decodability. As long as the number of receive antennas is greater than or equal to the number of users in the above sub-network, we can eliminate the interference of User m on the outgoing
symbols of User 1. Then, the outputs of different cancellation steps can be added to make the final equivalent channel representation of the User 1’s symbols and to decode them. A similar technique is applicable for decoding other users’ symbols. Following the elaborations in [31], the above technique can be generalized to a larger number of transmit antennas at each user in a fairly straightforward manner. However, we cannot go through more details for brevity.

V. NUMERICAL RESULTS

In Table I, we show the number of extra complex multiplications that PODs and PSTBCs require for decoding each symbol, compared to the counterpart open-loop STBCs. According to these numbers, PODs require fewer operations at the decoder. Also in Table I, the quantization computational complexities of PODs and PSTBCs are listed in the cases that we addressed in our simulations. PODs require far fewer operations at the quantization step compared to those of PSTBCs. As a rule of thumb, the number of complex multiplications for implementing the quantizer of PSTBCs scales as $N M M_t M_r$, whereas this number scales as $N \lambda M_r$ for PODs that use $\lambda$-dimensional general vector codebooks. Note that when we use a vector codebook similar to (8), the number of complex multiplications at the quantizer scales as $N M_r$.

In the first set of simulation results, in Fig. 3, we plot the Bit Error Rate (BER) performance of different transmission schemes that we studied in this paper over an $8 \times 1$ MISO channel. With $N = 2$ and $N = 4$ feedback regions, the best PSTBCs are the ones that use $8 \times 4$ and $8 \times 2$ precoders, respectively, and Alamouti inner codes. The $\lambda = 2$ and $\lambda = 4$ PODs in these scenarios perform inferior to the corresponding PSTBCs. However, they incur lower decoding and quantization complexities. With $N = 6$, the situation is different. The $\lambda = 6$ POD slightly outperforms the best PSTBC, the one with $8 \times 2$ precoders, although the POD’s decoding and quantization complexity is lower. Note that with $N = 8$ feedback regions, the $\lambda = 8$ POD, the directional beamforming (BF) and the PSTBC that uses $8 \times 1$ precoders are similar structures.

In Fig. 4, we show the performance of the best PODs and PSTBCs over a $4 \times 1$ MISO channel, when the feedback link carries one of $N = 3$ possible feedback indices. The feedback signaling in this case can be performed for example using a 3-PSK constellation. This figure shows the simulation results for both real and complex constellations. In this case, directional beamforming provides diversity order 3. Note that here, $\frac{M_t}{N}$ is not an integer and PODs with $\lambda = 3$-dimensional vector codebooks outperform the best PSTBCs, the ones which employ $4 \times 2$ precoder matrices. Moreover, the PODs have less complexity requirements.
The performance of the proposed design techniques for MIMO transmission is first examined over a $4 \times 2$ channel, as plotted in Fig. 5. Here, regardless of the number of receive antennas, the PSTBCs with $M_t \times M$ precoders provide the best error performance when $\frac{M_t}{M} = N$. However, when $\frac{M_t}{N}$ is not an integer, PSTBCs can be outperformed by $\lambda = N$ PODs. According to Table I, again the complexity of implementing PODs is lower than that of PSTBCs.

Finally, the performance of the most general example of this paper is depicted in Fig. 6. The code structures in this experiment are explained in Section III-C and are examined over a $5 \times 2$ MIMO channel. The values of the power loading parameters, reported in the example of Section III-C are optimized through this Monte Carlo simulation. Using this example, we again show that with vector codebooks and more noticeably, with matrix codebooks, PODs can outperform PSTBCs.

Our extensive simulations, some reported above, have shown that when $\frac{M_t}{N}$ is an integer, PODs cannot outperform a PSTBC with $M = \frac{M_t}{N}$ inner STBC code rows. However, in these cases, PODs with $\lambda = N$-dimensional beamformers are low-complexity alternatives that can provide performance results close to those of PSTBCs. When $\frac{M_t}{N}$ is not an integer, however, combining $\lambda = N$-dimensional beamforming and coding by PODs is superior to PSTBCs. Note that in these cases, the best performance results are attributed to $M_t \times T$ PODs that use $\left( \lambda = \min\{M_t, NM \} \times M = \lfloor \frac{M_t}{N} \rfloor \right)$ precoders and added $M_t - \lambda$ length $T$ orthogonal rows. Note also that when $\frac{M_t}{N} \leq 1$, the best transmission strategy is directional beamforming.

The superiority of combining coding and precoding (or beamforming) by PODs to pure precoding by PSTBCs in the latter cases can be resulted from the packing properties of the precoder and beamformer codebooks with those cardinalities and dimensions [13] [24]. We leave further investigations of this matter for future work.

In the last experiment, we show the performance of the interference cancellation technique, we developed using PODs. In Fig. 7, the multi-user example in Section IV is simulated and the BER performances of the two users, one with beamforming and the other in open-loop mode are shown. Using only $M_r = 2$ receive antennas, the diversity order 2 is provided to both users with interference cancellation. Note that using this semi-closed-loop structure, not only does the BER performance of the user with beamforming (thus the average BER of the system) outperform the open-loop code from [31], but also the same single decodability feature is maintained. Based on the example of Section IV, one can develop other multi-user scenarios with closed-loop or open-loop users. Our primary results in this paper show that with as few receive antennas as the number of the users,
a diversity order equal to the number of transmit antennas on each user can be provided to them. Increasing the number of receive antennas beyond the number of users yields multiplicative diversity order in theory. With $M_r$ receive antennas we can obtain the diversity order $M_t(M_r - J + 1)$ [31]. In Fig. 7, we also show the performance of the system with $M_r = 3$ receive antennas, which in theory provides diversity order 4.

VI. CONCLUSIONS

In this paper, we presented the design of PODs, a general class of matrix constellations that create a larger solution space for combining space-time coding and beamforming, compared to conventional PSTBCs. Our focus was to establish low-complexity combining schemes in the decoding and quantization stages. Through pairwise error probability analysis, and based on high-resolution quantizer assumptions, we analytically optimized POD structures with vector beamformers over MISO channels. We also showed that the derived properties are extendable to transmission across MIMO channels and to the PODs with matrix precoders. It was shown that PSTBCs are a sub-class of PODs. If $\frac{M_t}{N}$ is not an integer, PODs outperform PSTBCs.

Based on the MIMO transmission schemes of this paper, we also constructed PODs applicable to multiple-antenna multi-user channels, to combine beamforming and space-time coding with spatial multiplexing. For this scenario, a very low-complexity interference cancellation and decoding technique was established that scales the diversity order of the system with the number of transmit and receive antennas.

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REFERENCES


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Fig. 1. System block diagram.

Fig. 2. Average CGM versus SNR for different beamforming dimensions $\lambda$, with $M_t = 6$ transmit antennas, and $N = 4$ feedback regions.
Fig. 3. 8x1 MISO channel, Rate = 1 bit/sec/Hz using BPSK symbols.
Fig. 4. 4x1 MISO channel, $N = 3$ feedback regions, Rate = 1 bit/sec/Hz using BPSK symbols and Rate = 2 bits/sec/Hz using QPSK symbols.
Fig. 5. 4x2 MIMO channel, Rate = 1 bit/sec/Hz using BPSK symbols.
Fig. 6. 5x2 MIMO channel, Rate = 1 bit/sec/Hz using BPSK symbols.
Fig. 7. 2-user interference cancellation, 2-transmit antennas per user and 2 and 3 receive antennas. User 1 can perform either beamforming or open-loop transmission. Rate = 4 bits/sec/Hz using QPSK symbols.