Communication Over Multiple Antenna Fading Channels Using Quantized and Erroneous Feedback Information

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University of California, Irvine
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To my beloved family:

Amir, Simin, and Sourena
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Expected rate maximization over slowly fading channels using erroneous quantized feedback
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Abstract of the Dissertation

Communication Over Multiple Antenna Fading Channels
Using Quantized and Erroneous Feedback Information

by
Siavash Ekbatani
Doctor of Philosophy in Electrical and Computer Engineering
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Professor Hamid Jafarkhani, Chair

We study communication schemes with a quasi-static wireless channel model. It means that over the duration of several symbol transmissions, the fading channel remains constant. Therefore, the receiver is capable of estimating the channel coefficients accurately through some pre-defined training data sequences before the real data communication takes place. Using only the receiver knowledge about the channel, open-loop communication is possible. However, the main question is that if we can improve these schemes using channel information at the transmitter with a reasonable amount of complexity and overhead.

In the most efficient class of closed-loop communications, transmitter knowledge of the channel conditions is obtained using a feedback link from the receiver to
the transmitter. The feedback information must be communicated over a reverse channel in an adequately reliable and quick fashion, noting that communication over each realization of the channel cannot begin before the feedback link is established. Moreover, the amount of communication resources, such as bandwidth and power dedicated to feedback information must be kept minimal in order not to significantly affect the rest of the network and to impose the least possible interferences on other users.

There is a trade-off between the feedback resources and reliability in the closed-loop systems. If we want to keep the feedback information scarce, low power and prompt, we cannot help dealing with the issue of unreliable feedback information. In this sense, forward links are easier to deal with than the feedback links. Because, it is reasonable for the receivers to wait for rather long time duration and decode the transmitted data with some delay, whereas, feedback must be communicated quickly to minimize the time wasted before the feedback link is established. Furthermore, the forward link carries more power resources and bandwidth than the feedback link. Therefore, our main focus in this study is to design reliable and high-rate communication schemes when the knowledge of the transmitter from the channel condition is very limited and also unreliable (error-prone).

Feedback errors and mismatches are extra sources of errors in a close-loop system. Therefore, if not handled suitably, they can degrade the performance of the system severely and make it even worse than an open-loop system. We show that through well-defined performance measures, we can take into account the feedback error properties in the system design procedure and adjust the system parameters such that we always benefit from feedback, regardless of how little or how erroneous the feedback information might be.
The work presented in this thesis can be broken into two major categories. In the first two chapters, we study fixed-rate and fixed-power practical communication schemes and the goal is to minimize the reception error probability using limited and noisy feedback. The last three chapters consider information theoretic aspects of wireless communications systems with variable communication rate and transmission power, where the design goals are to minimize the outage probability and the distortion of the system and to maximize the expected reliable transmission rate. In every chapter, we introduce an independent performance measure for the specific wireless communication application across multiple-antenna fading channels and provide design rules and methodologies to take advantage of feedback information, taking into account its limitations and reliability issues. The performance measures that we optimize in the sequel cover a vast scope, such as error rate of coded modulation, outage probability, expected distortion, and expected rate. In every separate system, the main goal is to present a design technique that can outperform the open-loop counterpart with any amount of feedback information and any feedback channel quality factor.
Chapter 1

Introduction

For communications over wireless fading channels, multiple-antenna transmitters and receivers provide significant performance improvements compared to single antennas. The main benefits of using multiple antennas are better reliability with lower error probability and higher achievable data rates. The main obstacles for reliable and high-rate communications are the limited resources that the transmitters and the receivers can spend in terms of the available power, bandwidth, and computational complexity. Therefore, a decent communication scheme should be designed while taking into account these factors. The transmission schemes designed and developed in this context depend on the channel conditions and the availability of receiver and transmitter information about the state of the channel.
1.1 Channel Information at the Receiver and the Transmitter Sides of Communications Links

1.1.1 Quasi-Static Fading Channels

According to the broad literature of wireless communications and therefore, the well-established standard models that are practically in use today, practical communication takes place across quasi-static channels. This channel model essentially states that the channel coefficients between the transmitter and the receiver sides of the communication links remains constant over the duration of a multitude of transmission symbols and then it changes independently afterwards [32, 93]. This duration of time is called the channel coherence time and every occurrence of a fading channel coefficient is called a fading block. From the data transmission perspective, we communicate one data frame over every fading block.

In a practical system, since the channel remains constant during a frame of data, the receivers can estimate the channel fading coefficients (channel state information or CSI) through training. We call the receiver side channel state information as CSIR. Using accurate CSIR, an antenna array at the receiver can adjust the gains and the phases of received signals at different antenna elements to maximize the desired reliability measure. A communication scheme that only relies on CSIR is called an open-loop scheme since only supports the forward link transmission. Among all open-loop communication schemes, space-time coding has attracted the most attention since it is a very reliable and low-complexity scheme [47].

On the other hand, it is truly desirable and useful if some sort of channel state information is available at the transmitter, which we call it CSIT. Using CSIT, the transmitter antenna array can also perform gain and phase adjustment and
improve the properties of the system. A communication system that uses CSIT is called a closed-loop communication scheme, since it should depend on a mechanism that conveys CSI from the receive side to the transmission side and therefore, it requires to also establish a backward communication link before data transmission takes place.

A closed-loop communication scheme can be realized through several means. In some applications, we use the concept of duality, which states that the forward link and the backward link are similar and one-side channel estimation can serve as CSIT for transmission through the other side. However, in most of the applications, different transmission sides occur at different times or frequencies and duality is not a practical assumption.

1.1.2 Statistical Channel State Information

Knowledge of the statistics of the channel fading processes can be also a useful piece of CSIT, such as the long-term mean vector or the long-term covariance matrix of the channel vectors. Note that the statistics don’t change as quickly as the coefficients do, and therefore, they can be more reliably conveyed to the transmitters. A transmitter antenna array can estimate the channel mean or covariance information using time domain estimation of the reverse link, or we can implement a non-frequent feedback channel in the system that carries very low rate information about these parameters once per several data frames [48, 67, 104, 105].

These two types of closed-loop schemes are specially suitable for correlated channels. The channel spatial correlation often degrades the error performance of the open-loop systems if it is not known a priori. However, if these pieces of information can be used as CSIT, they can even improve the performance of a
communication system. [50, 94]. It has been shown that over multiple antenna transmitters, with available channel mean information with a high quality estimates, single dimensional beamforming is optimal and for low quality estimate of the mean, multi-fold diversity through space-time coding is optimal [43, 94]. Also with available channel covariance information, multi-fold diversity transmission is optimal with a water filling type of power spreading based on the channel covariance [43,94]. It was also elegantly shown that by available channel side information to the transmitter, separating coding and beamforming doesn’t incur any capacity loss and achieves the highest possible data rate [85].

1.1.3 Instantaneous Channel State Information

Due to the reliability reasons, it is more attractive in practical standards to use instantaneous CSIT, such as in WLAN, WCDMA, WiMAX, 3G-LTE, and etc.. Namely, instead of relying on statistical information, we prefer to estimate the channel coefficients directly at the receiver and somehow inform the transmitter about the obtained complex channel coefficients. This proposal requires to use a feedback link to convey the channel state information from the receiver to the transmitter at every frame. This type of closed-loop communication schemes is the focus of our work and therefore, from now on, we only refer to (instantaneous) CSIT, even though we won’t mention in every occasion. We refer to the upcoming chapters for more details about instantaneous CSIT and its applications.
1.1.4 CSIT Limitations: Quantization and Reliability and Our Contributions

In a practical communication environment, we try to reduce the overhead of feedback. Not only for the sake of the subject link, but also to reduce the unintended interference on the other links of a communication network. Therefore, practical systems must use some sort of quantization at the receiver side to represent the CSIT efficiently using a limited number of bits. Then the quantization bits must be conveyed over a feedback link from the receiver to the transmitter in an independent manner from the forward link transmission.

Conventionally, a closed-loop communication scheme can be modeled using the block diagram of Fig. 1.1. If we can assume that obtaining perfect and error-free feedback information is practical, this block diagram is flawless. However, we can simply question this model noting the following facts: Feedback information must be carried through fading channels. Backward channel resources such as power and bandwidth are limited since in the majority of applications, the receiver units have low transmission powers. Also feedback information can only bear a short period of time due to its severe sensitivity to delays and since within the time of feedback transmission in a frame, the transmitter has to remain idle. All the above factors
Figure 1.2: A schematics of closed-loop communication with erroneous feedback result in feedback channel distortions that can be caused by quantization losses and feedback channel errors, i.e., due to the low rate and low reliability factors of the feedback links.

The limitations of the rate and the reliability of feedback channels open a vast area of research, which is still on-going in the communication community. In this thesis, we explore a portion of the open issues and provide some system design guidelines to combat the limitations of feedback channels. In the most generic form, the system block diagram of our interest can be shown in Fig. 1.2. We assume that feedback information is not ideal and it can be subject to errors and mismatches.

In a closed-loop communication system, if feedback information is carried with uncertainties, it can introduce a source of error to the whole system and can lead to severe performance degradation, if the system is not designed robustly. We will show that several conventional techniques in the literature can suffer severely from uncertainties of feedback information. Our contribution is to fundamentally revise these design procedures and make them robust against unreliable feedback. The main objective is to give the whole system enough flexibility, such that it can
adjust its parameters based on the error properties of the feedback links. This goal can be achieved if with any amount of feedback error probability, the system can outperform a no-CSIT scheme and its performance can approach that of an ideal closed-loop system when the feedback link error factor diminishes.

In some cases, designing robust communication schemes requires some additional information other than the CSIR at the receiver. In some transmission schemes, the receiver should know the transmitters perception of the CSI in the form of erroneous feedback information. Therefore, the system block diagram in Fig. 1.2 contains an additional link from the transmitter to the receiver, which is called a “genie channel”. Let us now emphasize that implementing the concept of genie is way more practical than error-free feedback for the following reasons: The amount of information that a genie channel carries is equal to the feedback information. Note that in all the practical communications schemes, the forward link carries several fields of control information and adding genie information doesn’t introduce much overhead. Also note that the transmitter sides carry much more transmission power than that of the receiver sides. Moreover, forward channel is not delay sensitive and limited in length and duration and by definition several data symbols must be transmitted over each fading block, before decoding at the receiver begins. Therefore, genie can be made simply as a part of channel estimation, or parallel to the data, or a part of the synchronization information in a practical communication link.

In the following chapters of the dissertation, our objective is to design several transmission schemes that can utilize limited bandwidth and unreliable feedback information in more efficient and robust ways that exist in the literature. In the rest of this chapter, we will discuss some preliminary concepts about the conven-
tional closed-loop systems. Then, in the following chapters, we go through the modifications needed to make them robust against the uncertainties of the feedback channels.

1.2 Multiple Antenna Beamforming for Practical Communication Using Partial CSIT

For practical wireless communications across quasi-static fading channels, multiple antenna arrays are attractive since they can provide desirable performance properties.

1.2.1 Brief Outline

The first two chapters of the thesis are dedicated to study practical aspects of multiple antenna wireless communications. The system design objective in this context is to minimize the error probability of detecting the received data. The transmission rate and the transmission power is fixed to minimize the implementation complexity of the system. In the following subsections, we discuss two main benefits of utilizing multiple antennas at the transmitter of a wireless communication system, i.e., the array gain and the diversity order.

1.2.2 Array Gain

The array gain is defined based on the average received signal power at the receiver sides.
Open-loop Uncoded System

Suppose that the transmitter is using $M_t$ antennas and the power budget is being split among them. In other words, each transmitter antenna sends out $\frac{1}{\sqrt{M_t}} x$, where $x$ denotes a scalar sequence of data symbols. Denoting the channel vector by the normal complex Gaussian vector $h = \begin{bmatrix} h[1] & \cdots & h[M_t] \end{bmatrix}^T$, the transmit receive signal relation can be modeled by

$$y = \left( \sum_{i=1}^{M_t} h[i] \right) \frac{1}{\sqrt{M_t}} x + n$$

(1.1)

Note that the fading coefficients $h[i] : i \in \{1, \cdots, M_t\}$ are modelled as complex Gaussian variables with variance $\frac{1}{2}$ per complex dimension $^1$. The signal $x$ is transmitted from all the $M_t$ transmit antennas. Hence, the average transmit power is $\overline{P_x} = \mathbb{E}_X\{|x|^2\}$ and the transmit power is fixed as $P_x^{\text{max}} = \max_x |x|^2$. The variable $\nu = \left( \sum_{i=1}^{M_t} h[i] \right)$ is zero-mean complex Gaussian with variance $\frac{M_t}{2}$ per complex dimension. Therefore, the received signal power is

$$P_y = \frac{1}{M_t} \mathbb{E}_\nu\{\nu^2\} \mathbb{E}_X\{|x|^2\} = \mathbb{E}_X\{|x|^2\}$$

(1.2)

and the array gain is 1.

Closed-loop System with Perfect CSIT

Now suppose that the transmitter knows the current realization of the channel perfectly. The optimal transmission scheme is “beamforming” with the transmitted signals formulated as

$$\hat{x} = \begin{bmatrix} h^*[1] \|h\|_F x & \cdots & h^*[M_t] \|h\|_F x \end{bmatrix}^T$$

(1.3)

\footnote{This model follows directly by applying the law of large numbers in a reach scattering environment [32].}
The average transmit power is $P_b = E_X \{|x|^2\}$ and the received signal can be shown as

$$\hat{y} = \left( \sum_{i=1}^{M_t} \frac{|h[i]|^2}{\|h\|_F^2} \right) x + n = \|h\|_F x + n$$  \hspace{1cm} (1.4)

Therefore, the received power can be formulated as follows:

$$P_y = E_h \{ \|h\|_F^2 \} E_X \{|x|^2\} = M_t E_X \{|x|^2\}$$  \hspace{1cm} (1.5)

As a result, the additional gain of signal-to-noise ratio (SNR) compared to the open-loop system, i.e. the array gain is $M_t$.

**Beamforming with High-Rate Quantized Direction Feedback**

High-rate quantized beamforming is the most popular beamforming scheme in the literature, which works based on the channel direction vector, $\hat{h} = \frac{h}{\|h\|_F}$. Suppose that we fix a vector codebook $\mathcal{W} = \{\omega_1, \omega_2, \cdots, \omega_K\}$ at the transmission ends. The receiver can decide about the optimal beamforming vector from the codebook $\mathcal{W}$ and conveys this decision to the transmitter using $r = \log_2 K$ feedback bits.

For the index decision at the receiver, the optimal decision strategy is as follows:

$$i = \arg \max_{i \in \{1, \cdots, K\}} \left| \omega_i^\dagger \hat{h} \right|^2$$  \hspace{1cm} (1.6)

Therefore, the quantizer Voronoi regions can be formulated as $^2$

$$\mathcal{V}_i = \left\{ h; \left| \omega_i^\dagger h \right|^2 \geq \left| \omega_j^\dagger h \right|^2 \right\}: \forall j \neq i$$  \hspace{1cm} (1.7)

and the receive signal equation can be written as

$$\hat{y} = \gamma \omega_i^\dagger h x + n$$  \hspace{1cm} (1.8)

$^2$A Voronoi region is a part of the source of quantization (that is the channel vector in this context) that must be represented by an index [29].
In order to calculate the average receive SNR, we can proceed as follows. The channel vector can be represented as a product of its amplitude and its direction, \( h = \gamma h \). The channel amplitude square \( \eta = \gamma^2 \) is Chi-square distributed with \( 2M_t \) degrees of freedom, i.e., with the density function \( P(\eta) = \frac{\eta^{M_t-1} \exp(-\eta)}{(M_t-1)!} : \eta \in [0, \infty] \).

In each quantization codebook, the Voronoi region associated to each codeword \( \omega_i \) can be expressed by \( S_{\omega_i}(z) = \{ g; 1 - |\omega_i^* g|^2 < z \} \) for \( i \in \{1, \cdots, 2^r\} \). Furthermore, the probability density function of the vector \( g \) in each Voronoi region can be mapped to a uniform distribution \( p_g_{\omega_i}(z) = U(S_{\omega_i}(\beta)) \). Here, we assume that the whole \( M_t \)-dimensional unit-magnitude complex number field can be partitioned as \( C^{M_t} = \bigcup_{i=1}^{2^r} S_{\omega_i}(\beta) \), which is more accurate for higher quantization rates, \( r_g \). Note that \( \beta = \max z \) is known as the conforming radius of each codebook. In \( S_{\omega_i}(z) \), let us denote the cumulative distribution function (c.d.f) of the random variable \( z = 1 - \max_{\omega_i} |\omega_i^* g|^2 \) by \( F_Z(z) \). It is known that \( A(S(z)) = \frac{2^{2M_t} z^{M_t-1}}{(M_t-1)!} \) is equal to the surface area of the spherical cap \( S(z) \). As a result, \( F_Z(z) \) can be tightly upper bounded by

\[
F_Z(z) < \tilde{F}_Z(z) = \sum_{i=1}^{K} \frac{A(S_{\omega_i}(z))}{A(S(z))} = \begin{cases} \gamma z^{M_t-1} & 0 \leq z < \beta \\ 1 & z \geq \beta \end{cases}
\]

(1.9)

where \( \gamma = \beta^{-(M_t-1)} \). Here, the conforming radius is \( \beta = 2^{\frac{-r_g}{M_t-1}} \) without making the Voronoi regions overlap.

It is straightforward to see that the average receive SNR can be formulated as

\[
\tilde{P}_y = \mathbb{E}_\eta \{ \eta \} \mathbb{E}_h \{ |\omega_i h|^2 \} \mathbb{E}_x \{ |x|^2 \}
\]

(1.10)

Note that \( \mathbb{E}_\eta \{ \eta \} = M_t \) and

\[
\mathbb{E}_h \{ |\omega_i h|^2 \} \leq 1 - \left( 1 - \frac{1}{M_t} \right) 2^{-\frac{-r_g}{M_t-1}}
\]

(1.11)
Therefore, by employing the quantized direction feedback and performing beamforming, the achievable array gain can be expressed by:

\[ M_t - (M_t - 1) 2^{-\frac{r}{M_t-1}} \]  \hspace{1cm} (1.12)

Therefore, the achievable array gain using a rate-limited feedback link is less than that of a perfect CSIT scheme, \( M_t \).

### 1.2.3 Diversity Order

The diversity order is defined based on the transmission error probability and is an important measure of reliability of a wireless communication system. Mathematically, it is the decay rate of the error probability term with respect to the transmission power, at very high SNR.

#### Open-loop Uncoded System

Consider the same transmission system description that was mentioned in the previous section. For the BPSK signalling with maximum likelihood (ML) decoding algorithm, the transmit receive signal relation can be modeled through

\[ y = \left( \sum_{i=1}^{M_t} h[i] \right) \frac{1}{\sqrt{M_t}} x + n, \]

where the noise term \( n \) is complex Gaussian with variance \( \frac{M_t}{2\text{SNR}} \) per complex dimension. The conditional pairwise error probability (PEP) of decoding in favor of the erroneous symbol \( e \), when the symbol \( c \) is transmitted can be expressed as

\[ P \left( x \rightarrow e \mid h \right) = Q \left( \sqrt{\frac{\eta \text{SNR}}{2M_t}} |c - e|^2 \right) \]  \hspace{1cm} (1.13)

where the transmit signal power is assumed fixed in the BPSK signalling and the instantaneous SNR is defined by,

\[ \eta = \frac{1}{\sqrt{M_t}} \left| \sum_{i=1}^{M_t} h[i] \right|^2 \]  \hspace{1cm} (1.14)
The PEP expression in Equation (1.21) can be upper bounded tightly in high-SNR scenarios by a Chernoff bound:

\[
P(x \rightarrow e|h) \leq \exp \left( -\frac{\eta \text{SNR}}{4M_t}|c - e|^2 \right)
\]

(1.15)

The variable \( \eta \) follows an exponential distribution \( P(\eta) = \exp(-\eta) : \eta \in [0, \infty] \).

Therefore, the PEP can be derived as follows:

\[
P(x \rightarrow e) \leq \int_{0}^{\infty} \exp \left( -\eta \left[ 1 + \frac{\text{SNR}|c - e|^2}{4M_t} \right] \right) d\eta
\]

\[
= \frac{1}{1 + \frac{\text{SNR}|c - e|^2}{4M_t}}
\]

(1.16)

Let us recall the definition of the diversity order:

\[
d = -\lim_{\text{SNR} \rightarrow \infty} \frac{\log_{10}[P(x \rightarrow e)]}{\log_{10}[\text{SNR}]}
\]

(1.17)

According to the above definition, the diversity of the aforementioned transmission scheme is \( d = 1 \) and this open-loop system cannot be considered to be reliable.\(^3\)

Closed-loop System with Perfect CSIT

Now, let us study the real benefits of CSIT. With beamforming, the received signal can be formulated as \( \hat{y} = \|h\|_F x + n \), where the noise term \( n \) is complex Gaussian with variance \( \frac{M_t}{2\text{SNR}} \) per complex dimension. The conditional pairwise error probability (PEP) can be upper bounded by \( P(x \rightarrow e|h) \leq \exp \left( -\frac{\eta \text{SNR}}{4M_t}|c - e|^2 \right) \), where the variable \( \eta \) follows a Chi-square random variable with \( 2M_t \) degrees of freedom

\[
P(\eta) = \frac{\eta^{M_t-1} \exp (-\eta)}{(M_t - 1)!} : \eta \in [0, \infty]
\]

(1.18)

\(^3\)It is well-known that the most reliable open-loop transmission strategy is space-time coding (STC). For studying properties of STC schemes, we refer to [47], Chapter 2. STCs cannot provide any array gain. However, they yield \( M_t \)-fold diversity order per receive antenna.
Therefore, the PEP can be derived as follows:

\[
P(x \rightarrow e) \leq \int_0^\infty \frac{\eta^{M_t-1}}{(M_t - 1)!} \exp \left( -\eta \left[ 1 + \frac{\text{SNR}|c - e|^2}{4M_t} \right] \right) \, d\eta \tag{1.19}
\]

Noting that \( \Gamma[M_t] = \int_{t=0}^{\infty} \frac{t^{M_t-1} \exp(-t)}{(M_t-1)!} \, dt \), and by the replacement \( t = \eta \lambda \), the above integral can be simplified to

\[
P(x \rightarrow e) \leq \left[ 1 + \frac{\text{SNR}|c - e|^2}{4M_t} \right]^{-M_t} \tag{1.20}
\]

Let us recall the definition of the diversity order, \( d = -\lim_{\text{SNR} \to \infty} \frac{\log_{10}[P(x \rightarrow e)]}{\log_{10}[\text{SNR}]} \).

According to the above definition, the diversity of the aforementioned transmission scheme is \( d = M_t \) and the gain of diversity order is \( M_t \)-fold.

**Beamforming with High-Rate Quantized Direction Feedback**

This time, the received signal can be formulated by \( \hat{y} = \gamma \omega^\dagger_i h \, x + n \), where the noise term \( n \) is complex Gaussian with variance \( \frac{M_t}{2\text{SNR}} \) per complex dimension. Moreover, the conditional pairwise error probability (PEP) of decoding in favor of the erroneous symbol \( e \), when the symbol \( c \) is transmitted can be expressed as

\[
P(x \rightarrow e| h) = Q \left( \sqrt{\frac{\eta |\omega^\dagger_i h|^2 \text{SNR}}{2M_t}} |c - e|^2 \right) \tag{1.21}
\]

where the transmit signal power is assumed fixed in the BPSK signalling and \( \eta = \gamma^2 \).

Using the approximate derivation of the channel direction density function in Equation (2.23), the PEP expression in Equation (1.21) can be upper bounded by the following Chernoff bound:

\[
P(x \rightarrow e| h) \leq \exp \left( -\frac{\eta (1 - z) \text{SNR}}{4M_t} |c - e|^2 \right) \tag{1.22}
\]
The variable $\eta$ is distributed as
\[ P(\eta) = \frac{\eta^{M_t-1} \exp(-\eta)}{(M_t-1)!} : \eta \in [0, \infty] \] (1.23)
and distribution of the variable $z$ can be written as
\[ p(z) = K(M_t - 1)z^{M_t-2} : z \in [0, 2^{-\frac{r}{M_t-1}}] \] (1.24)

Therefore, the unconditional error probability can be upper bounded by
\[
P(x \rightarrow e) \leq \int_0^{2^{-\frac{r}{M_t-1}}} dz \int_0^{\infty} d\eta \exp \left( -\eta \left[ 1 + \frac{(1-z) \text{SNR} |c-e|^2}{4M_t} \right] \right) \frac{\eta^{M_t-1}}{(M_t-2)!} K z^{M_t-2}
\] (1.25)

Noting that $\Gamma[M_t] = \int_0^{\infty} \frac{t^{M_t-1} \exp(-t)}{(M_t-1)!} dt$, and by change of variable $\eta \lambda = x$, the above integral can be simplified to
\[
P(x \rightarrow e) \leq \int_0^{2^{-\frac{r}{M_t-1}}} dz \left( \frac{1}{1 + \frac{(1-z) \text{SNR} |c-e|^2}{4M_t}} \right)^{M_t} (M_t - 1)z^{M_t-2}
\] (1.26)

Let us use the change of variable
\[
v = 1 + \frac{(1-z) \text{SNR} |c-e|^2}{4M_t} \quad \Rightarrow \quad z = 1 - \frac{v-1}{u} \quad \Rightarrow \quad dz = -\frac{1}{u} dv
\]

Therefore we have
\[
P(x \rightarrow e) \leq \frac{(M_t - 1)}{u^{M_t-1}} \int_{1+u}^{1+ \left( \frac{1-2^{\frac{r}{M_t-1}}}{} \right) u} dv \left( \frac{u + 1 - v}{v^{M_t}} \right)^{M_t-2}
\]

Let us now solve $I = \int -dv \left( \frac{u+1-v}{v^{M_t}} \right)^{M_t-2}$ first. By the replacement $\frac{1}{u} + 1 - z = x$,
\[
I = \int \left[ \frac{u + 1 - v}{v} - 1 \right]^{M_t-2} \left( \frac{-1}{v} \right)^2 dv
\]
\[ P (x \rightarrow e) \leq \frac{1}{u^{M_t-1}} \frac{1}{u+1} \left[ \frac{u+1}{u} - 1 \right]^{M_t-1} \left( 1 + \left( 1 - 2^{-\frac{1}{M_t-1}} \right) u \right) \]

Therefore, the unconditional PEP can be simplified to the following expression:

\[ P (x \rightarrow e) \leq \frac{1}{u^{M_t-1}} \frac{1}{u+1} \left[ \frac{u+1}{u} - 1 \right]^{M_t-1} \cdot \frac{1+\left( 1 - 2^{-\frac{1}{M_t-1}} \right) u}{1+u} \]

where \( u = \frac{\text{SNR}}{4M_t} |c - e|^2 \).

According to the above definition, the diversity of the aforementioned transmission scheme is \( d = M_t \) and an \( M_t \)-fold diversity order can be achieved.

### 1.2.4 Multiple Receive Antenna Perspective

Apparently, by employing \( M_r \) receive antennas, the receiver can sample the received signal \( M_r \) times. Therefore, the average received power is \( M_r \) times the one with employing single receive antenna. If CSIR is perfect and flawless, employing multiple receive antennas is similar to transmitting the signal \( M_r \) times through independent channel realizations. Therefore, an \( M_r \)-fold additional receive diversity order is also achievable.

### 1.3 Power and Rate Adaptation Over Multiple Antenna Fading Channels Using Partial CSIT

#### 1.3.1 Brief Outline

The focus of the last three chapters of the thesis is on theoretical performance bounds with different design objectives. We will discuss outage minimization, distortion minimization, and rate maximization over a multiple-antenna wireless
system, when partial CSIT through a noisy feedback channel is available. Therefore, first in this section, we give a brief introduction of the previous work in the literature on design and analysis with error-free feedback assumptions.

1.3.2 Fixed-Rate Systems

In the simplest sense, suppose that we are going to communicate with a fixed rate and our objective is to minimize the outage probability of the system using power control based on partial CSIT. A $M_t$-antenna transmitter scales a rate $R$ nats per channel use $M_t$-dimensional complex Gaussian codeword $C$ by the CSIT-dependent power loading factor $\sqrt{P_i/M_t}$. This power control parameter is dictated by the receiver via the feedback index $i$. The covariance matrix of the transmission codeword is $\mathbb{E}\{CC^\dagger\} = I_{M_t}$, the identity matrix, where $\mathbb{E}\{\cdot\}$ denotes ensemble expectation and $\dagger$ represents the complex conjugate transpose operation. The elements of the received signal in each channel codeword can be represented by

$$y(l) = Hx(l) + n(l) \quad \forall l \in \{1, \cdots, K\}$$ (1.28)

where $l$ denotes the time index within a block of length $K$ channel uses. Using a scaled identity covariance codeword with transmission power $P$, the mutual information between the transmitter and the receiver data can be expressed as

$$I(P) \overset{\triangle}{=} \min_{\{r,t\}} \sum_{\kappa=1}^{\min\{M_r, M_t\}} \log \left(1 + \lambda_{\kappa} \frac{P}{M_t}\right)$$ (1.29)

where $\{\lambda_{\kappa}\}$, $k = 1, 2, \cdots, \min\{M_r, M_t\}$ denote the eigenvalues of $HH^\dagger$ [53]. For future references, let us define $P_R(H)$ to be the solution of

$$I(P) - R = 0$$ (1.30)
In other words, $\mathcal{P}_R(H)$ is the minimum power level required at the transmitter to invert the channel or the required transmission power so that the receiver can successfully decode the codeword of rate $R$.

### 1.3.3 Quantizer Design for Power Control

Suppose that the transmitter and the receiver employ a finite-sized power control codebook $\{\mathcal{P}_0, \cdots, \mathcal{P}_{K-1}\}$. The receiver chooses a member of this codebook based on the realization of the channel and conveys its index back to the transmitter using $\lceil \log_2 K \rceil$ feedback bit. The optimal quantizer structure to enable power control in the system and minimize the outage probability is the following:

$$
\mathcal{J}(H) = \begin{cases} 
  j & \text{if } \mathcal{P}_R(H) \in (\mathcal{P}_{j-1}, \mathcal{P}_j] \\
  0 & \text{if } \mathcal{P}_R(H) > \mathcal{P}_{K-1}
\end{cases}
$$

(1.31)

Here, the power constraint at the transmitter is expressed as $\mathbb{E}_H \{ \mathcal{P}_J(H) \} \leq \text{SNR}$.

To proceed with the optimization process, let us define the complimentary outage probability at rate $R$ and with the transmission power $\mathcal{P}$ as

$$
\hat{F}(\mathcal{P}) \triangleq \Pr \left[ I(\mathcal{P}) \geq R \right]
$$

(1.32)

Using this definition, the minimum outage probability of the proposed system can be expressed as

$$
\mathcal{P}_{\text{out}}^{\text{opt}} = 1 - \hat{F}(\mathcal{P}_{K-1})
$$

(1.33)

and the minimum power to achieve $\mathcal{P}_{\text{out}}^{\text{opt}}$ can be derived as

$$
\mathcal{P}_{\text{min}} = [F(\mathcal{P}_0) + 1 - F(\mathcal{P}_{K-1})]P_0 + \sum_{i=1}^{K-1} [F(\mathcal{P}_i) - F(\mathcal{P}_{i-1})]P_i
$$

(1.34)

The outage minimization problem can be therefore expressed as follows:

$$
\max \quad \mathcal{P}_{K-1}
$$

s.t. \quad $\mathcal{P}_{\text{min}} \leq \text{SNR}$

(1.35)
Solving the above optimization yields the optimal outage probability of the system [53].

1.3.4 Asymptotic Analysis of Outage Probability

Diversity gain can be defined as the decay rate of the outage probability with respect to the SNR:

\[ d = \lim_{\text{SNR} \to \infty} \frac{-\log P_{\text{out}}}{\log \text{SNR}} \]  

(1.36)

It is shown in [53] that with error-free feedback and for a fixed-rate system, the diversity gain is derived in the following theorem:

**Theorem 1.** The maximum diversity gain of a system with a power codebook of size \( K \) is

\[ d_K = \sum_{k=0}^{K-1} (d_0)^{k+1} \]

(1.37)

where \( d_0 = M_r M_t \) [53–56, 59].

In other words, if feedback is error-free, the diversity gain increases polynomially with the cardinality of the power control codebook.

1.3.5 Variable Rate Systems

Now consider a more complicated scenario, where the transmission rate is also adaptive with CSIT. Our design objectives in this context are two-fold: minimizing the average distortion with joint source-channel coding and maximizing the expected transmission rate. In the latter case, we also study the effects of smart coding schemes such as superposition coding. To introduce the design ideas behind power and rate adaptation with partial CSIT, in this section, we discuss the distor-
tion minimization problem with error-free CSIT in details. The rate maximization problem can be treated almost similarly.

In an information theoretic sense, a codeword with rate $R_i = \log(1 + \gamma_i P_i)$ and an allocated power of $P_i$ is reliably transmitted if the transmitter receives feedback index $i$ and $\gamma \geq \gamma_i$, where $\gamma$ is the magnitude of the channel realization vector. The probability of this event is $\omega_i \triangleq P(\gamma \geq \gamma_i, i)$ and the associated distortion is $D(bR_i)$, where $D(bR_i)$ is the Distortion-Rate function of a Gaussian source and $b$ is the bandwidth expansions ratio between source and channel coding.

An outage occurs with probability $P(\gamma < \gamma_i, i) = P(i) - \omega_i$. The expected distortion is then

$$E_D = \sum_{i=0}^{K-1} \left\{ \omega_i D(bR_i) + [P(i) - \omega_i] D_0 \right\}$$

(1.38)

where by defining the CDF of $\gamma$ by $F(\gamma)$, we can show that

$$\omega_i = p_{ij} \left[ F(\gamma_b^{i+1}) - F(\gamma_i) \right] + \sum_{j=i+1}^{K-1} p_{ij} \left[ F(\gamma_b^{j+1}) - F(\gamma_j) \right]$$

(1.39)

where $\{\gamma_b^i\}$ is the set of boundaries and $\{\gamma_i\}$ is the set of reconstruction points of the quantizer. Moreover, the transmit power constraint of this system can be expressed as

$$E_P = \sum_{i=0}^{K-1} v_i P_i \leq P$$

(1.40)

where

$$v_i \triangleq P(i) = p'_{i0} F(\gamma_0) + p_{i0} \left[ F(\gamma_b^i) - F(\gamma_0) \right] + \sum_{j=1}^{K-1} p_{ij} \left[ F(\gamma_b^{j+1}) - F(\gamma_j) \right]$$

(1.41)
1.3.6 Quantizer Design for Rate and Power Adaptation

In [20,21], the authors showed that the expected distortion and the expected power equations are then given as

$$\mathcal{E}_D = \sum_{i=0}^{K-1} [F(\gamma_{i+1}) - F(\gamma_i)] \ D(bR_i) + F(\gamma_0) \ D_0 \quad (1.42)$$

$$\mathcal{E}_P = \sum_{i=1}^{K-1} [F(\gamma_{i+1}) - F(\gamma_i)] \ P_i + F(\gamma_1) \ P_0 \quad (1.43)$$

and the he expected distortion minimization problem can be stated as

$$\min_{\{\gamma_i, P_i\}} \mathcal{E}_D \quad (1.44)$$

s.t. $\gamma_i \leq \gamma_{i+1}, \ P_i \geq 0$ and $\mathcal{E}_P \leq P$

1.3.7 Asymptotic Analysis of Distortion

Here, a similar concept to the diversity gain is important that is called the distortion exponent, $\delta$. It is is defined as the high-SNR decay rate of the expected distortion with respect to the SNR [62]

$$\delta = - \lim_{P \to \infty} \frac{\log \mathcal{E}_D}{\log P} \quad (1.45)$$

**Theorem 2.** Assuming noiseless feedback, a Gaussian source and a short-term power constraint, the distortion exponent of the proposed feedback scheme is given as

$$\delta = M_r \left(1 - \frac{1 - \frac{b}{M_r}}{1 - \left(\frac{b}{M_r}\right)^{K+1}}\right) \quad (1.46)$$

that is the same as the distortion exponent of a system with no CSIT that employs $K$-layer superposition coding [35] $^4$.

$^4$Since the multiple transmit antenna channels cannot be considered as degraded, we only discussed the results with multiple receive antennas. However, if we accept sub-optimal transmission strategies, for multiple transmit antenna transmission, we can split the power among different spatial directions, similar to the fixed-rate system case. In this case the distortion exponent can be modified by multiplying $M_t$ by the system dimension.
1.4 The Outline of the Thesis

The following chapters essentially discuss similar ideas to what we saw in the previous sections. However, in every case study, we revise the design criteria, methodologies, and algorithms to efficiently distance ourselves from an ideal feedback assumption and consider the effect of feedback errors. The goal of the entire design solutions is to define flexible system parameters, such that the system can adjust itself to the error properties of feedback channels and always outperform a no-CSIT scheme.

In Chapters 2 and 3, we study closed-loop communication over multiple antenna channels when quantized and noisy quantized feedback information are available to the transmitter. These design techniques are suitable for the practical communication systems, where the transmission rate and power are fixed to reduce the implementation complexity. The objective is to minimize the transmission error probability. We suggest various general structures to combine space-time coding and transmit beamforming. The goal of combining is to provide full diversity order and also additional array gain compared to the open-loop systems, with any feedback rate and reliability.

Chapters 4, 5, and 6 study the information theoretic aspects of the closed-loop systems with the most efficient available feedback schemes in the literature that currently only work when the feedback channel is error-free. The objective is to assign rate and power of the transmission codewords to optimize the desired performance criterion. We will show that by accounting for the error effects of the feedback channel in the design process, we can outperform open-loop systems with any feedback reliability factor. This is possible by proper CSI quantizer design adjustments with every particular feedback channel model.
Chapter 2

Combining Beamforming and Space-Time Coding Using Quantized Feedback

For multiple-antenna transmission over quasi static fading channels, we assume that the receiver is able to estimate the channel coefficients perfectly. Furthermore, there is a feedback link carrying quantized channel state information (CSI) from the receiver to the transmitter. With available partial CSI at the transmitter (CSIT), directional beamforming (or precoding) can increase the average received signal-to-noise ratio (SNR). This benefit of beamforming is called the array gain. It improves the capacity of communication systems [43,76], reduces the outage probability [99], and enhances the error performance. Partial CSIT is widely used in the literature to adjust the shape of the transmission matrix constellations in response to the channel, e.g. using precoded space-time block codes (PSTBCs) [48,67,82,104,105], antenna subset selection [34], and the combination of beamforming and power
In quasi-static environments, the channel remains constant in each data frame and changes independently afterwards. Therefore, the transmitter must acquire the CSI using a feedback link and update it in every frame. The main obstacle for this type of closed-loop communications is the bandwidth limitation of the feedback link, and the best treatment is to use quantization to reduce the feedback rate [70, 75, 81]. Many researchers have proven that if the feedback is error-free and delay-free, by combining space-time coding (STC) and beamforming (or precoding), obtaining the benefits of diversity order and array gain is possible with any feedback rate [1, 34, 64, 71, 74, 101]. To the best of our knowledge, however, the best performance results with rate-limited feedback are achieved by PSTBCs in [72]. These PSTBCs manipulate orthogonal space-time block codes (OSTBCs) in [47,89] interactively and make the transmit signals coherent to the channel direction, introduced by the feedback link.

We propose a new coded modulation class called Partly Orthogonal Designs (PODs). PODs have a space-time coding part which imposes low decoding complexity at the receiver due to its orthogonal structure. Also they have a beamforming or precoding part that uses quantized channel feedback. The goal of
the CSI quantization scheme used in the receiver/transmitter of a POD system is to maximize the received SNR (array gain). For this purpose, Grassmannian beamforming and unitary precoding codebook structures in the literature are two efficient and practical schemes [70, 72, 98]. Designing PODs is also inspired from these beamforming strategies. Nonetheless, PODs are more general combining structures compared to PSTBCs and in fact PSTBCs can be viewed as a sub-class of PODs. In contrast to the previously known PSTBCs, PODs perform the beamforming operation inside the code matrices. In many cases, the members of the POD matrix constellation class require less decoding complexity and much less quantization complexity compared to the conventional PSTBCs. Some PODs also outperform PSTBCs as we will discuss in this following sections.

We also extend POD structures for transmission over multiple-antenna multi-user channels. A multiple-access scenario is considered, where some users employ beamforming and some users work in the open-loop mode. In this case, from the receiver’s point of view, the aggregation of the transmitted signals from different users makes a “virtual” POD constellation matrix. In this case, a POD can be considered as a way to combine spatial multiplexing, space-time coding, and transmit beamforming.

2.0.1 Organization and Notation

The rest of this chapter is organized as follows: In Section 2.1, we introduce the components of the system, including PODs with different configurations, feedback channel and quantizer specifications, and decoder structures. In Section 2.2, we analyze the performance of PODs and optimize these code structures based on a pairwise error probability metric. Extending the scope of PODs to the multi-user
channels is presented in Section 2.3. In Section 2.4, several simulation results are provided to show the properties of PODs in different configurations. Finally, we draw our conclusions in Section 2.5.

In this chapter, $\|x\|_F$ denotes the Frobenius norm of the matrix or vector $x$. The operators $(\cdot)^T$, $(\cdot)^\dagger$, and $(\cdot)^*$ stand for transpose, Hermitian, and conjugate, respectively. The inner product of two vectors of the same size is shown by $\langle X, Y \rangle$. Finally, $p(x)$ and $E_x\{f(x)\}$ show the probability density function (pdf) of a random variable and the expectation of a function with respect to that variable, respectively.

## 2.1 System Components

As shown in Fig. 4.1, our transmission scheme consists of a multiple-antenna point-to-point wireless forward link with $M_t$ transmit antennas and $M_r$ receive antennas. The input bit stream is mapped to the baseband $M_t \times T$ modulation symbol matrices, where $T$ denotes the duration of each constellation matrix in time. We normalize the power of each transmission matrix to $M_t T$ and we assume that the transmission power is constant.

The channel remains constant over the duration of multiple matrix symbols. At the receiver, ideal channel estimation is assumed. The receiver also uses vector quantization to generate a feedback index, representing the state of the channel in each quasi-static fading block. The feedback channel is assumed to be error-free and delay-free. Therefore, the transmitter chooses its symbol constellation (a set of matrices to be used for modulation) based on the feedback index generated at the receiver and combines coding and beamforming. Upon receiving the signal,
knowing the feedback symbol and thus the transmission symbol constellation, the receiver performs maximum likelihood (ML) decoding.

2.1.1 Partly Orthogonal Designs (PODs)

In the most generic form, we build PODs on the rows of an orthogonal space-time block code (OSTBC) [89]. Suppose that the transmitter employs $M_t$ transmit antennas and the goal is to combine a $N$-dimensional directional (vector) beamformer $\omega_i$ with an OSTBC. To construct a POD, we proceed the following steps:

- Pick an OSTBC matrix constellation of size $M \times T$, where $M \geq M_t - N + 1$.
- Keep the first $M_t - N$ rows of the OSTBC at their places. These rows construct the non-beamformed part of the POD.
- Multiply each element of the row $M_t - N + 1$ by the $N$-dimensional beamforming vector $\omega_i$, where $\|\omega_i\|^2_F = 1$ and place the resulting column vectors at the bottom of the code matrix. The resulting sub-matrix constructs the beamformed part of the POD.
- Multiply the non-beamformed part of the code by the real power loading factor $\nu$ and the beamformed part by the real power loading factor $\mu$, where $\mu^2 + (M_t - N)\nu^2 = M_t$ is the instantaneous power constraint at the transmitter.
For example, we can use the following orthogonal design matrix from a real constellation, \{z_i\} \[47]:

\[
Z = \begin{bmatrix}
z_1 & z_2 & z_3 & z_4 \\
z_2 & -z_1 & z_4 & -z_3 \\
z_3 & -z_4 & -z_1 & z_2 \\
z_4 & z_3 & -z_2 & -z_1
\end{bmatrix}
\] (2.1)

Using a 2-dimensional beamformer \(N = 2\), the following structure is a POD:

\[
Z_2 = \begin{bmatrix}
\nu z_1 & \nu z_2 & \nu z_3 & \nu z_4 \\
\nu z_2 & -\nu z_1 & \nu z_4 & -\nu z_3 \\
\mu z_3 \omega_i & -\mu z_4 \omega_i & -\mu z_1 \omega_i & \mu z_2 \omega_i
\end{bmatrix}
\] (2.2)

where \(w_i = [w_i[1] \ w_i[2]]^T\) is the beamforming vector chosen by the feedback index. Following the same procedure, we can perform 3 and 4-dimensional beamforming for PODs as follows:

\[
Z_3 = \begin{bmatrix}
\nu z_1 & \nu z_2 & \nu z_3 & \nu z_4 \\
\mu z_2 \omega_i & -\mu z_1 \omega_i & \mu z_4 \omega_i & -\mu z_3 \omega_i
\end{bmatrix}
\] (2.3)

\[
Z_4 = \begin{bmatrix}
\mu z_1 \omega_i & \mu z_2 \omega_i & \mu z_3 \omega_i & \mu z_4 \omega_i
\end{bmatrix}
\] (2.4)

Later on we will show that the above matrix constellations enable separate ML decoding of symbols. After defining the POD design criteria, we will also clarify how to choose \(N, \mu,\) and \(\nu\). Note that the code structure in (3.4) is a PSTBC which employs vector codebooks, i.e., where the inner code is \(1 \times T\) and the precoder matrix (vector) is \(M_t \times 1\).

One can build a POD starting from bottom to top. It is sufficient to find \(M_t - N + 1\) orthogonal rows of a STBC matrix, keep \(M_t - N\) rows at their places, and multiply each element of the last row by \(\omega_i\). With some \(M_t\) and \(N\) dimensions,
we cannot find $M_t - N + 1$ complex orthogonal rows. In these cases one can also extend the idea of PODs to partly quasi-orthogonal designs (PQODs) using a quasi-orthogonal STBC (QOSTBC) structure \cite{46,95}. Remember that QOSTBCs are pairwise decodable and so are their counterparts in the PQOD class.

### 2.1.2 Feedback Channel and Quantizer

The wireless channel model that we consider in this context is quasi-static, zero-mean, circular symmetric, and complex Gaussian with independent and identically distributed (i.i.d.) coefficients. We represent this multiple-antenna channel by an $M_t \times M_r$ matrix $\hat{h}$. Suppose that the $N \times M_r$ channel sub-matrix $\tilde{h}$ from channel matrix $\hat{h}$ is processed at the receiver and the index of a vector quantizer (VQ) encoder is conveyed back to the transmitter. The sub-matrix $\tilde{h}$ is composed of the last $N$ rows of $\hat{h}$, i.e.,

$$\hat{h} = \begin{bmatrix} \tilde{h}_{(M_t-N)\times M_r}^T & \tilde{h}_{N\times M_r}^T \end{bmatrix}^T \quad (2.5)$$

We quantize $N$ rows of $\hat{h}$ in the matrix $\tilde{h}$ where $N \leq M_t$.

In our system, the transmitter uses a constant power. In this case, the knowledge of the magnitude of $\tilde{h}$, $\|\tilde{h}\|_F$ cannot help the transmitter to improve the performance of the system. We instead use the direction of $\tilde{h}$, denoted by $\bar{h} = \tilde{h}/\|\tilde{h}\|_F$. The magnitude and the direction of $\hat{h}$ are independent and our quantization target is to encode $\bar{h}$ as a source matrix. Note that the order of rows in POD code matrices is not important. Therefore, we can arbitrarily change the indices of the channel rows that are subjects of quantization.

To encode $\bar{h}$, a $N$-dimensional quantization vector codebook $\mathcal{W} = \{\omega_1, \cdots, \omega_K\}$ with cardinality $K$ is defined at the transmitter/receiver ends. To select from $\mathcal{W}$, the feedback link carries $r = \log_2(K)$ information bits. The transmitter adjusts
the transmission constellation matrix based on this feedback index.

We use a conventional quantization metric aiming to maximize the average received SNR, by choosing $\omega_i \in \mathcal{W}$, and finding the pair of Voronoi regions $\mathcal{V} = \{\mathcal{V}_1, \cdots, \mathcal{V}_K\}$ and reconstruction codebook $\mathcal{W} = \{\omega_1, \cdots, \omega_K\}$, such that \cite{70,81}

$$\left( \mathcal{V}, \mathcal{W} \right) = \arg \max_{\mathcal{V}, \mathcal{W}} \mathbb{E}_{\mathbf{h} \in \mathcal{V}_i} \left\{ \left\| \omega_i^\dagger \mathbf{h} \right\|_F^2 \right\}$$

(2.6)

for $i \in \{1, \cdots, K\}$. Each Voronoi region is delineated by

$$\mathcal{V}_i = \left\{ \mathbf{h} \in \mathbb{C}^{N \times M_r}; \text{s.t.,} \left\| \omega_i^\dagger \mathbf{h} \right\|^2_F \geq \left\| \omega_j^\dagger \mathbf{h} \right\|^2_F \right\}$$

(2.7)

for $\forall i, j \in \{1, \cdots, K\}$, where $\mathbb{C}^{N \times M_r}$ is the field of unit-norm $N \times M_r$ complex matrices. Moreover, the centroids (members of the reconstruction codebook) can be derived as

$$\omega_j = \arg \max_{\omega \in \mathbb{C}^N} \mathbb{E}_{\mathbf{h} \in \mathcal{V}_i} \left\{ \left\| \omega^\dagger \mathbf{h} \right\|_F^2 \right\}$$

(2.8)

In an i.i.d. channel environment, the optimal quantization codebook as the solution to the latter maximization problem can be obtained by Grassmannian line packing \cite{70}, also expressed as

$$\mathcal{W} = \arg \min_{\mathcal{W} \in \mathbb{C}^N} \max_{i,j \in \{1, \cdots, K\}} \left| \omega_i^\dagger \omega_j \right|$$

(2.9)

In a special case, where $K \leq N$, with a slight abuse of the notation, a simple codebook that satisfies (2.9) is the following:

$$\mathcal{W} = \left\{ \begin{array}{c}
\omega_1 = \left[ \begin{array}{c}
1 & 0 & \cdots & 0
\end{array} \right]^T \\
\omega_2 = \left[ \begin{array}{c}
0 & 1 & \cdots & 0
\end{array} \right]^T \\
\vdots \\
\omega_K = \left[ \begin{array}{c}
0 & \cdots & 1 & 0
\end{array} \right]^T
\end{array} \right.$$
Using the above codebook for directional beamforming is equivalent to antenna selection among $K$ antennas. When $K > N$, however, the codebook needs to be designed explicitly. In this case, we can maximize the average received SNR using (2.6). Alternatively, we can design the quantizer codebook based on solving (2.9). A simple solution to this problem is through the unitary constellation design technique in [72] (in the special case $M_t = N$ and $M = 1$, where $M$ is the number of columns of the precoder matrix or the row dimension of the inner STC in a PSTBC).

After designing the quantizer, the quantizer encoder at the receiver must solve the maximization $i = \arg \max_i \| \omega_i^\dagger \hat{h} \|_F^2$ and the quantizer decoder at the transmitter uses a look-up table based on the feedback index $i$. The quantization computational complexity at the receiver scales linearly with the source row dimension, $N$.

### 2.1.3 Decoding PODs

We use an example to show the decoding complexity of PODs with vector codebooks. In the code structures (3.3)-(3.4), the rows are orthogonal to each other, meaning that their inner product is zero. This property translates to separate ML decoding of symbols. For example, suppose that $Z_2$ from (3.3) is transmitted over the fading channel $\hat{h}$. The received vector is

$$\hat{r} = \begin{bmatrix} \nu z_1 & \nu z_2 & \nu z_3 & \nu z_4 \\ \nu z_2 & -\nu z_1 & \nu z_4 & -\nu z_3 \\ \mu z_3 & -\mu z_4 & -\mu z_1 & \mu z_2 \\ \mu z_3 & -\mu z_4 & -\mu z_1 & \mu z_2 \end{bmatrix} \begin{bmatrix} \nu z_1 \\ -\nu z_1 \\ \nu z_4 \\ -\nu z_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \hat{h} + \bar{n} \quad (2.11)$$
where in the MISO case \( \hat{r} = [r_1 \ r_2 \ r_3 \ r_4]^T \), \( \hat{h} = [h_1 \ h_2 \ h_3 \ h_4]^T \), and \( \pi = [n_1 \ n_2 \ n_3 \ n_4]^T \). For decoding symbol \( z_1 \), it is sufficient that the decoder constructs

\[
r_1 h_1^* - r_2 h_2^* - r_3^* \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} \omega_i^* =
\]

\[
z_1 \left( \nu |h_1|^2 + \nu |h_2|^2 + \mu^2 \left| \left\langle \omega_i, \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} \right\rangle \right|^2 \right) + n_1'
\]

(2.12)

Here, \( \langle \cdot, \cdot \rangle \) denotes the inner product and \( n_1' \) denotes the equivalent noise variable. Note that the decoding of \( z_1 \) is independent from that of \( z_2, z_3, \) and \( z_4 \). Similar arguments can be applied to the decoding of \( z_2 \). To decode \( z_3 \), the decoder needs to construct

\[
r_1 \begin{bmatrix} h_3^* \\ h_4^* \end{bmatrix} \omega_i + r_3^* h_1 - r_4^* h_2 =
\]

\[
z_3 \left( \nu |h_1|^2 + \nu |h_2|^2 + \mu^2 \left| \left\langle \omega_i, \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} \right\rangle \right|^2 \right) + n_3'
\]

(2.13)

Therefore, \( z_3 \) is decoded independently from \( z_1, z_2, \) and \( z_4 \). The decoding of \( z_4 \) is similar to \( z_3 \).

The only added operation compared to the decoding of open-loop OSTBCs is to calculate the product \( \begin{bmatrix} h_3 \\ h_4 \end{bmatrix} \omega_i^* \). This operation requires at most \( N \) complex multiplications, when we use one receive antenna. If \( \omega_i \) contains one element equal to 1 and all others equal to 0, like in (2.10), this operation does not require any complex multiplications.

### 2.1.4 PODs with Matrix Codebooks

The original PSTBCs in [72] use \( M \times T \) OSTBC constellation matrices as inner codes and multiply them by an \( M_i \times M \) precoder matrix \( P_i \) that is a member of
the precoding matrix codebook $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_K\}$, with cardinality $K$. The objective of the precoder codebook design is again to maximize the average received SNR. The practical codebook design strategy of [72] borrows constellation matrices from [38]. For arbitrary precoder dimensions in general, this technique results in a specific precoder structure that performs well with any feedback codebook cardinality (feedback rate). The complexity of codebook design exponentially increases with the number of transmit antennas and the number of feedback regions.

Love, et al. proved that if $K \geq \frac{M_t}{M}$ and the columns of the precoding codebook $\mathcal{P}$ (when all the $M_t \times M$ members are stacked in a matrix) span $\mathbb{C}^{M_t}$, then the PSTBC provides full-diversity order.

Inspired from PSTBCs, we can build a POD using matrix precoders instead of vector beamformers. The new construction algorithm is similar to the previous one. Suppose that the goal is to combine a $N \times M$-dimensional matrix precoder $\mathcal{P}_i$ with an OSTBC, when $N \leq M_t$. We proceed the following steps to design a POD:

• Pick an OSTBC matrix constellation of size $(M_t - N + M) \times T$.

• Keep the first $M_t - N$ rows of the OSTBC at their places. These rows construct the non-precoded part of the POD.

• Multiply the remaining sub-matrix of $M$ rows by the $N \times M$ precoder matrix $\mathcal{P}_i$ and place the result at the bottom of the code matrix. The resulting rows construct the precoded part of the POD.

• Multiply the non-precoded part by the real power loading factor $\nu$ and the precoded part by the real number $\mu$. 
The above structure degenerates to a PSTBC when $M_t = N$. Overall, in PODs coding and beamforming or precoding are combined. However, there is no separate coding part in PSTBCs. In this sense, PODs provide more solutions for combining coding and beamforming compared to PSTBCs. We can find some cases where PODs outperform PSTBCs, as will be shown in Section 2.4.

Decoding PSTBCs is more complex than that of open-loop OSTBCs. The added complexity is to calculate the equivalent channel matrix by constructing $P_i^\dagger \hat{h}$ at the receiver. The number of complex multiplications to calculate the equivalent channel scales with the precoder dimension. For example, with $M_t \times M$ precoders, this operation requires $M_t M$ complex multiplications. For PODs with matrix codebooks, the same equivalent channel calculation is required with $N \times M$ precoder matrices. Therefore, the number of additional operations scales with $N M$. Note that compared to decoding PODs with vector codebooks, we need more operations since in the case of PODs with vector codebooks, the added complexity is at most $N$ complex multiplications. Note also that in MIMO applications, the number of additional complex operations is counted per receive antenna.

### 2.2 Performance Analysis

Using pairwise error probability analysis, we analytically optimize POD structures for transmission over MISO channels. We also analytically extend the proof of full-diversity order for transmission over MIMO channels with matrix or vector codebooks. Then we numerically optimize the generally extended PODs with different antenna configurations using Monte Carlo simulations.
2.2.1 Analysis and Optimization of PODs with Vector Codebooks Over MISO Channels

Upon transmitting the POD matrix $Z$, the $T \times 1$ received signal can be modeled as

$$ r = Z^\dagger \hat{h} + \pi $$  \hspace{1cm} (2.14)

where $\pi$ is the complex and additive white Gaussian noise with variance $\sigma^2_n/2 = M_t/(2 \text{ SNR})$ on the real and imaginary parts. Based on our normalization assumption, the SNR is defined as the ratio of the transmission power over the average noise power per receive antenna.

The conditional PEP of decoding in favor of $Z'$ when $Z$ is transmitted can be expressed as

$$ P(Z \rightarrow Z'|\hat{h}) = Q\left( \sqrt{\frac{1}{2\sigma^2_n}} \| \hat{h}^\dagger (Z - Z') \|_F \right) $$  \hspace{1cm} (2.15)

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left( -\frac{y^2}{2} \right) dy$ is the Gaussian tail function [47]. This expression can be tightly upper bounded in moderate to high SNR values by the Chernoff bound

$$ P(Z \rightarrow Z'|\hat{h}) \leq \frac{1}{2} \exp \left\{ -\frac{d^2(Z, Z')}{4\sigma^2_n} \right\} $$  \hspace{1cm} (2.16)

where $d^2(Z, Z') = \hat{h}^\dagger (Z - Z')(Z - Z')^\dagger \hat{h}$.

Suppose that the receiver provides the transmitter with $\log_2(K)$ feedback bits to represent the direction of $\bar{h}$ using the vector codebook $W = \{\omega_1, \cdots, \omega_K\}$. We decompose $\bar{h}$ into its magnitude and its direction through $\bar{h} = \sqrt{\rho} \bar{\pi}$, where $\rho = \sum_{i=1}^{N} \bar{h}[i]^2$ is the magnitude square and $\bar{\pi}$ is the direction. With a zero-mean complex Gaussian channel, the vector $\bar{\pi}$ is uniformly distributed over the unit-norm complex sphere $C^N$, [75], and $\rho$ is a Chi-square random variable with $2N$ degrees of freedom, characterized by $p(\rho) = \rho^{N-1}e^{-\rho}/(N - 1)!$. 
The PEP, conditioned on the feedback information can be expressed as

\[
P(Z \rightarrow Z') = E_{\tilde{h} | W} \left\{ \frac{1}{2} \exp \left( -\frac{1}{4\sigma_n^2} \left[ \tilde{h}^\dagger \tilde{h} \right] \right) (Z - Z')(Z - Z')^\dagger \right\}
\]

(2.17)

where the sub-script \( \tilde{h} | W \) reflects the feedback operation. For PODs from constellation \( \{z_i\} \), by some straightforward manipulations, the above expression can be expanded as

\[
\text{CGM} \overset{\Delta}{=} P(Z \rightarrow Z') = E_{\tilde{h}} \left\{ \frac{1}{2} \exp \left( -\Delta_z \left[ \mu^2 \rho \zeta + \nu^2 \|\tilde{h}\|_F^2 \right] \right) \right\}
\]

(2.18)

where \( \zeta = \max_{i \in \{1, \ldots, K\}} \|\langle \tilde{h}, \omega_i \rangle\|^2 \) and \( \Delta_z = \left( \frac{\sum_{i=1}^K |z_i - z'_i|^2}{4\sigma_n^2} \right) \) is proportional to the inner codes Euclidian distance and the SNR. Equation (3.9) is the definition of coding gain metric (CGM) in our analysis and design approach. To proceed, we write the CGM in an integral form:

\[
\text{CGM} = \frac{1}{2} \int_{\mathcal{R}^+} \int_{\Phi_{\tilde{h}}} \int_{\Phi_\zeta} \exp \left( -\Delta_z \left[ \mu^2 \rho \zeta + \nu^2 \|\tilde{h}\|_F^2 \right] \right) p(\rho, \tilde{h}, \zeta) \, d\tilde{h}
\]

(2.19)

Here, \( \mathcal{R}^+ \) means non-negative real and \( \Phi_{\tilde{h}} \) is the domain for the un-quantized portion of the channel vector, and \( \Phi_\zeta \) is the domain of \( \zeta \).

For independent and identically distributed (i.i.d.) channel realizations, the unquantized part of the channel, \( \tilde{h} \) is independent from the quantized part, \( \bar{h} \). Furthermore, the channel direction, \( \bar{h} \) is independent from its magnitude, \( \rho \). Therefore, \( p(\rho, \tilde{h}, \bar{h}) \, d\tilde{h} = p(\tilde{h}) \, p(\zeta) \, p(\rho) \, d\tilde{h} \, d\zeta \, d\rho \).

**Lemma 1:** Using quantized vector beamforming, when \( N \leq K \), the CGM in (2.19) can be approximated by

\[
\frac{1}{1 + \mu^2 \Delta_z} \left( \frac{1}{1 + (\mu^2 \Delta_z)(1 - \beta)} \right)^{N-1} \left( \frac{1}{1 + \nu^2 \Delta_z} \right)^{M-N}
\]

(2.20)
where $\beta = 2^{-\log_2(K)}$ is a specification of the reconstruction codebook $W$, as defined in the proof.

Proof: To quantify CGM in (2.19), first we should derive the pdf of

$$\zeta = \max_{i \in \{1, \ldots, K\}} |\langle \mathbf{h}, \mathbf{w}_i \rangle|^2$$ (2.21)

To facilitate the derivations, when $K \geq N$, we adopt a high-resolution quantizer assumption [28]. Based on this assumption for our source vector $\mathbf{h}$, all the Voronoi regions can be characterized by spherical caps around the reconstruction vector endpoints $\omega_i$, where the all-zero vector is the origin. The intuition behind this assumption is that with $K \geq N$ non-overlapping conical regions in a $N$-dimensional space, we can approximately tile the whole unit-norm hyper-sphere, $C^N$.\footnote{In contrast to this scenario, when $K < N$, the optimal quantization codebook is:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}^T, \ldots, \begin{bmatrix} 0 & 0 & \cdots & 1 & 0 \end{bmatrix}^T$$

With the above codebook, the tail of the quantization subject, $\mathbf{h}$, does not affect the quantization metric. Therefore, non-overlapping Voronoi regions that can tile $C^N$ cannot be defined.}

The spatial gaps as the result of this tiling diminish as the quantization rate increases.

The latter spherical cap regions can be parameterized by [106]

$$S_{\omega_i}(z) = \left\{ \mathbf{h} \in C^N : 1 - \left| \omega_i^\dag \mathbf{h} \right|^2 \leq z \right\}$$ (2.22)

where we call $0 < z < 1$ the chordal radius [70]. Furthermore, the whole $N$-dimensional unit-norm complex sphere $C^N$ can be spanned by $\bigcup_{i=1}^{K} S_{\omega_i}(z)$. Equation (2.7) suggests that $\zeta$ defined in (2.21) can characterize $V_i$. We can alternatively consider the cumulative distribution function of $z$ from (2.22) denoted by $F_Z(z)$ that characterizes $S_{\omega_i}(z)$. Using the surface area of the spherical cap region $S(z)$ in [75], $A(S(z)) = 2\pi^N z^{N-1}/(N-1)!$, it is shown in [106] that

$$F_Z(z) < \widetilde{F}_Z(z) = \sum_{i=1}^{K} \frac{A(S_{\omega_i}(z))}{A(S(z))} = \begin{cases} \gamma z^{N-1} & : 0 \leq z < \beta \\ 1 & : \beta \leq z \geq 1 \end{cases}$$ (2.23)
where $\beta = \left(1 - \max_{i,j} \left| \omega_i^\dagger \omega_j \right| \right) / 2$ is called the conforming radius and equals to the largest value of $z$ that does not make overlapping Voronoi regions, and $\gamma = \beta^{-(N-1)}$. For large $K$, the conforming radius is $\beta = 2^{-\log_2(K)}$ [75, 106].

Assuming that the inequality in (2.23) is tight, the CGM in (3.9) can be expressed as

$$\text{CGM} \approx \frac{1}{2} \int_0^\infty d\rho \int_0^1 d\tilde{F}_Z(z) \int_0^\infty d\vartheta \frac{\rho^{(N-1)} e^{-\rho} (N-1)!}{p_\rho(\vartheta) \exp(-\mu^2 \Delta_z \rho(1-z)) \exp(-\nu^2 \Delta_z \vartheta)}$$

(2.24)

In the above integral, $\vartheta = \|\hat{h}\|_F^2$ follows a Chi-square distribution with $2(M_t - N)$ degrees of freedom. Note that from [47]-(3.19) we have $\int_0^\infty p_\rho(\vartheta) \exp(-\nu^2 \Delta_z \vartheta) d\vartheta = (1 + \nu^2 \Delta_z)^{-(M_t - N)}$.

To simplify the presentation, let us use the following auxiliary variable:

$$\mathcal{I} = \frac{1}{(N-1)!} \int_0^1 d\tilde{F}_Z(z) \int_0^\infty d\rho \exp\left(-\frac{1 + \mu^2 \Delta_z (1-z)}{\xi} \rho \right) \rho^{(N-1)}$$

Note that $\int_0^\infty \exp(-\xi \rho) \rho^{N-1} d\rho = \frac{(N-1)!}{\xi^N}$. Therefore, we have

$$\mathcal{I} = \frac{1}{2} \int_0^1 \frac{d\tilde{F}_Z(z)}{[1 + \mu^2 \Delta_z (1-z)]^N} < \frac{1}{2} \int_0^1 \frac{d\tilde{F}_Z(z)}{[1 + \mu^2 \Delta_z (1-z)]^N}$$

(2.25)

where the inequality can be confirmed by integration by part. By a simple change of variable $1 + \mu^2 \Delta_z (1-z) = u$, the left hand side of (2.25) can be reexpressed as

$$\mathcal{I} = \frac{1}{2} \int_0^\beta \gamma(N-1) z^{N-2} dz = \frac{\gamma}{2} \frac{1 + \mu^2 \Delta_z (1-\beta)}{1 + (\mu^2 \Delta_z) (1-\beta)}^{N-1}$$

(2.26)

which shows that the overall CGM can be approximated by Equation (2.20).

We use the result of Lemma 1 to find the optimal row dimension of the beamforming part, $N$, and the optimal power loading factors $\mu$ and $\nu$ in POD structures.
Later on, using numerical analysis, we will confirm these results through Monte Carlo simulations.

**Theorem 1:** Based on the CGM expression in (2.20), we can prove the following properties:

1. When $N \leq K$, PODs are full-diversity order constellations.

2. The PEP performance of PODs is improved by increasing the feedback rate.

3. In the high SNR region, the optimal power allocation strategy is to spend the same amount of power on the beamformed part and the non-beamformed parts of PODs, i.e., $\mu = \sqrt{N}$.

**Proof:** In the high SNR region, we can reformulate the CGM expression in (2.20) as

$$
\text{CGM} = \frac{1}{2} (\Delta z \mu^2)^{-1} \left[ \Delta z (1 - \beta) \mu^2 \right]^{-N+1} \left[ \Delta z \nu^2 \right]^{-M_t + N}
$$

Note that the instantaneous power constraint at the transmitter yields $\mu^2 + (M_t - N) \nu^2 = M_t$. Therefore, at high-SNR we have

$$
\text{CGM} = \frac{1}{2} \Delta z^{-M_t} (1 - \beta)^{-N+1} \left( \frac{\mu^2}{M_t - \mu^2} \right)^{-M_t + N}
$$

(2.27)

The first property is clear from the definition of the diversity order.

$$
g_d = - \lim_{\Delta z \to \infty} \frac{\log[P(Z \to Z' | i)]}{\log[\Delta z]} = M_t
$$

(2.28)

The second property can be proven by noting the role of the conforming radius $\beta$ in the CGM expression. As the number of quantization regions increases, $\beta$ becomes smaller. Therefore CGM decreases and POD constellations provide better performance.
To prove the third property we investigate the behavior of CGM with respect to $\mu^2$. Minimizing CGM is a simple constraint optimization. First we calculate the first-order derivative of CGM with respect to $\mu^2$, i.e.,

$$\frac{d(CGM)}{d(\mu^2)} = K \left[ -N \left( \mu^2 \right)^{N-1} \left( \frac{M_t - \mu^2}{M_t - N} \right)^{-M_t+N} + \left( \mu^2 \right)^{-N} \left( \frac{M_t - \mu^2}{M_t - N} \right)^{-M_t+N-1} \right]$$

(2.29)

where $K$ is a constant. Note that $\mu^2 \neq 0$. Further, we assume that $N < M_t$. Because the solution for $N = M_t$ is obviously $\mu^2 = N = M_t$ and $\nu^2 = 0$. Moreover, we ignore the case $\mu^2 = M_t$ since it only associates to $N = M_t$. Setting the first-order derivative equal to zero yields $-N (M_t - \mu^2)/(M_t - N) + \mu^2 = 0$. The solution to the above equation and the power constraint is $\mu^2 = N$ and $\nu^2 = 1$.

It is straightforward to show that the second-order derivative of the CGM is always positive. So the CGM is convex with respect to $\mu^2$ and the latter power loading strategy is globally optimal. □

The performance of PODs is improved by increasing the beamforming dimension, as long as $N \leq K$. This is shown in Fig. 2.2 by plotting CGM versus SNR from (2.20) for different parameters $N$. Fig. 2.2 depicts these plots for an $M_t = 6$ antenna system, when $K = 4$ feedback regions are used. From this figure we conclude that minimizing the PEP requires us to choose $N = \min\{M_t, K\}$. By this choice in a POD structure, we obtain full-diversity order and minimum pairwise error probability. Fig. 2.2 also shows the optimality of the third property in Theorem 1 in the finite SNR regime. We deviated $\mu$ slightly from $\sqrt{N}$ and observed that the CGM increases. We can see that even in low SNR regions CGM is globally minimum when $\mu = \sqrt{N}$.
2.2.2 Analysis of PODs With Matrix or Vector Codebooks

Over MIMO Channels

The following theorem defines the relationship between the design parameters $N$, $M$, and $K$ in a full-diversity system.

**Theorem 2:** If $K \geq \frac{N}{M}$ and the columns of the precoder codebook $\mathcal{P}$ span $\mathbb{C}^N$, then the PODs that use this codebook provide full-diversity order.

**Proof:** With a slight abuse of the notation, let $\mathcal{Q}_i = \omega_i$ with vector beamformers or $\mathcal{Q}_i = \mathcal{P}_i$ with matrix precoders. For a POD structure that uses $N$-dimensional beamformer vectors $\omega_i$ or $N \times M$ precoder matrices $\mathcal{P}_i$ to quantize $\bar{t}$, we define
the coding gain metric, CGM, as

\[
\mathbb{E}_{\tilde{h}|W}\left\{ \frac{1}{2} \exp\left( -\left[ \tilde{h}_{(M_t-N)\times M_r}^\dagger \tilde{h}_{N\times M_r}^\dagger \right] \frac{(\mathbf{Z}-\mathbf{Z}')(\mathbf{Z}-\mathbf{Z}')^\dagger}{4\sigma_n^2} \right) \right\}
\]

Using our quantization scheme, every Voronoi region can be also defined as

\[
\mathcal{V}_i = \left\{ \tilde{h}; \text{s.t.,} \quad \left\| Q_i^\dagger \tilde{h} \right\|_F \geq \left\| Q_j^\dagger \tilde{h} \right\|_F \right\} : \forall i, j \in \{1, \cdots, K\}
\]

With i.i.d. channels, \( \tilde{h} \) and \( h \) are independent and the CGM expression can be expanded as

\[
\text{CGM} \overset{\Delta}{=} \frac{1}{2} \mathbb{E}_{\tilde{h}} \left\{ \exp \left( -\mu^2 \Delta_z \left\{ \max_{i \in \{1, \cdots, K\}} \left\| Q_i^\dagger \tilde{h} \right\|_F^2 \right) \right\}
\]

\[
\times \mathbb{E}_{\tilde{h}} \left\{ \exp \left( -\nu^2 \Delta_z \left\| \tilde{h} \right\|_F^2 \right) \right\}
\]

\[
= \frac{1}{2} \left( \mathbb{E}_{\tilde{h}} \left\{ \exp \left( -\mu^2 \Delta_z \left\{ \max_{i \in \{1, \cdots, K\}} \left\| Q_i^\dagger \tilde{h} \right\|_F^2 \right) \right\} \right) \right)^{\frac{1}{(M_t-N)\times M_r}}
\]

\[
\times \left( 1 + \nu^2 \Delta_z \right)^{(M_t-N)\times M_r}
\]

Note that in PODs with vector codebooks, \( M = 1 \). We also mentioned that \( N \leq K \). Moreover, in PODs with matrix codebooks \( \frac{N}{M} \leq K \). Inspired from [72]-Theorem 1, we know that

\[
-\lim_{\Delta_z \to \infty} \frac{\log \left[ \text{CGM} \right]}{\log \left[ \Delta_z \right]} = NM_r.
\]

Therefore, it is easy to see that

\[
g_d = -\lim_{\Delta_z \to \infty} \frac{\log \left[ \text{CGM} \right]}{\log \left[ \Delta_z \right]} = M_tM_r
\]

and therefore, PODs provide full-diversity order.

Note that using unitary constellations borrowed from [38] for precoding satisfies the assumption of this theorem.

We can also show the following properties of PODs by Monte Carlo simulations:
• Using $N$-dimensional vector beamformers with $M_t - N$ non-beamformed rows, the optimal power loading parameters are $\mu^2 = N$ and $\nu^2 = 1$. For this structure, the best value of the beamformer dimension is $N = \min\{M_t, K\}$.

• Using $N \times M$ precoders with $M_t - N$ non-precoded rows, the optimal power loading parameters are $\mu^2 = \frac{N}{M}$ and $\nu^2 = 1$. With this structure, if $M_t > K$, the best dimension of the precoder matrix is $N \times M$, with the number of inner code rows $M = \lfloor \frac{M_t}{K} \rfloor$, the largest integer smaller than $\frac{M_t}{K}$ and with the spatial dimension $N = \min\{M_t, KM\}$.

2.2.3 More POD Examples

In this example, we design PODs with both vector and matrix codebooks. In Section 2.4, we use this example to compare the performance of different POD configurations. Suppose that the transmitter employs $M_t = 5$ antennas and the feedback codebook defines $K = 2$ quantization regions. Furthermore, suppose that we use BPSK signalling and real orthogonal design as the inner code. A conventional PSTBC uses a $3 \times 4$ orthogonal inner code and a $5 \times 3$ precoder codebook $\mathcal{P} = \{P_1, P_2\}$:

$$Z_{\text{PSTBC}} = \sqrt{\frac{5}{3}} P_i \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_3 & -z_4 & -z_1 & z_2 \\ z_2 & -z_1 & z_4 & -z_3 \end{bmatrix}$$ (2.33)

The scalar coefficient $\mu = \sqrt{\frac{5}{3}}$ is chosen to satisfy the power constraint, assuming that the precoder codebook is built as in [72]. Using a vector codebook $\mathcal{W} = \{\omega_1, \omega_2\}$ of cardinality 2, we pick the beamforming dimension $N = 2$ and use the
following POD structure:

\[ Z_{\text{POD}_{\text{vec}}} = \begin{bmatrix}
  z_1 & z_2 & z_3 & z_4 \\
  z_3 & -z_4 & -z_1 & z_2 \\
  z_2 & -z_1 & z_4 & -z_3 \\
  \sqrt{2} z_2 \omega_i & -\sqrt{2} z_1 \omega_i & \sqrt{2} z_4 \omega_i & -\sqrt{2} z_3 \omega_i \\
\end{bmatrix} \]  

(2.34)

where the optimality of \( \mu = \sqrt{2} \) is verified numerically. The POD structure in (2.34) outperforms the PSTBC in (2.33). Note that in this case, \( \frac{M_t}{M} < K \). We can alternatively build a POD with a matrix codebook \( \mathcal{P} = \{ P_1, P_2 \} \), using \( M \times T = 2 \times 4 \) OSTBC inner codes and \( N \times M = 4 \times 2 \) precoders and leaving \((M_t - N) \times T = 1 \times 4\) dimensions without precoding:

\[ Z_{\text{POD}_{\text{matrix}}} = \begin{bmatrix}
  z_4 & z_3 & -z_2 & -z_1 \\
  \sqrt{\frac{4}{2}} P_i \begin{bmatrix}
  z_1 & z_2 & z_3 & z_4 \\
  z_2 & -z_1 & z_4 & -z_3 \\
\end{bmatrix} \\
\end{bmatrix} \]  

(2.35)

In this case we can show numerically that \( \mu = \sqrt{\frac{4}{2}} \) minimizes the error rate. In Section 2.4, we show that the POD in (2.35) outperforms the other structures shown in (2.33)-(2.34).

### 2.3 Extension to Multiple-Antenna Multi-User Channels

Based on the application of PODs in transmission over MIMO channels, we use PODs for multi-user (multiple-access) communications. We consider transmission in the uplink of a wireless network, where the base station unit decodes independent information symbols from different users using multiple receive antennas.
We assume that some users in the network can employ beamforming and some users can only transmit in the open-loop mode. Normally, the major difficulty for implementing such a system is the large complexity of maximum-likelihood decoding. Inspired from the design methodologies in [51, 90], we develop a low-complexity joint interference cancellation technique that provides the users with a diversity order that scales with the number of transmit antennas at each user and the number of receive antennas at the access point.

We describe our design technique using an example. Suppose that there are $J = 2$ users, each employing $M_t = 2$ transmit antennas. User 2 can employ beamforming, whereas User 1 can only transmit in the open-loop mode. In this example we use QPSK signalling and we assume that the number of feedback regions for User 2 is $K = 2$. Therefore, User 2 uses the vector codebook $\mathcal{W} = \{\omega_1, \omega_2\}$ for beamforming. According to the discussions in the previous sections, an SNR maximizing codebook for this case is $\mathcal{W} = \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 \end{bmatrix}^T \right\}$.

In [90], the authors provided an open-loop transmission and decoding technique using Alamouti STBCs, which was simplified and used in [51] for larger systems. We extend these design techniques to our multi-user beamforming scenario using PODs. The transmit signals from both users form the following POD constellation:

$$Z_{MU} = \begin{bmatrix}
-z_1(1) & z_2(1)^* \\
z_2(1) & z_1(1)^* \\
\cdots & \cdots \\
-\sqrt{2}z_1(2)\omega_i & \sqrt{2}z_2(2)^*\omega_i
\end{bmatrix} \quad (2.36)$$

where the numbers in the parentheses denote the index of the users and the columns are indexed by time. User 1 transmits the upper sub-matrix and User 2 transmits
the lower sub-matrix of (2.36). The receiver is capable of estimating the channel

\[ \hat{h} = \begin{bmatrix} \tilde{h} \\ \ldots \\ \bar{h} \end{bmatrix} = \begin{bmatrix} h_{1,1}(1) & h_{1,m}(1) \\ h_{2,1}(1) & h_{2,m}(1) \\ \vdots & \vdots \\ h_{1,1}(2) & h_{1,m}(2) \\ h_{2,1}(2) & h_{2,m}(2) \end{bmatrix} \]  

(2.37)

where the first index denotes the transmit antenna and the second one denotes the receive antenna. We have shown the second receive antenna by \( m \) so that we can describe the decoding algorithm for any number of receive antennas. The code structure in (2.36) in fact combines beamforming, space-time coding, and spatial multiplexing. Because it transmits 4 symbols from 2 independent antenna subsets with rate 2 symbols per time slot. Note that since PSTBCs combine all the code rows together, they cannot be used to transmit independent symbols from different antennas. In other words, PSTBCs cannot be used for beamforming over multi-user channels. The constellation matrix in (2.36) belongs to a sub-class of PODs, where we have a smaller number of columns compared to the PODs in (3.3)-(3.4) and other similar point-to-point examples.

The encoder of the quantizer at the receiver solves \( i = \arg \max_{i=\{1,2\}} \| \omega_i \hat{h} \|_F \) to find the feedback index. Moreover, the received signal at the base station can be expressed as

\[ \begin{bmatrix} r_{1,1} & r_{1,m} \\ r_{2,1} & r_{2,m} \end{bmatrix} = \mathcal{Z}_{MU}^\dagger \hat{h} + \begin{bmatrix} n_{1,1} & n_{1,m} \\ n_{2,1} & n_{2,m} \end{bmatrix} \]  

(2.38)

In order to decode the transmit signals from Users 1 and 2, the receiver constructs
the following equivalent channel matrices:

\[
H_1(1) = \begin{bmatrix}
-h_{1,1}(1)^* & h_{2,1}(1)^* \\
h_{2,1}(1) & h_{1,1}(1)
\end{bmatrix}
\]

(2.39)

\[
H_m(1) = \begin{bmatrix}
-h_{1,m}(1)^* & h_{2,m}(1)^* \\
h_{2,m}(1) & h_{1,m}(1)
\end{bmatrix}
\]

and the following auxiliary matrices:

\[
\Omega_1(2) = \text{Diag} \begin{bmatrix}
-\sqrt{2} \left[ \omega_i(1)h_{1,1}(2)^* + \omega_i(2)h_{2,1}(2)^* \right] \\
\sqrt{2} \left[ \omega_i(1)^*h_{1,1}(2) + \omega_i(2)^*h_{2,1}(2) \right]
\end{bmatrix}
\]

(2.40)

\[
\Omega_m(2) = \text{Diag} \begin{bmatrix}
-\sqrt{2} \left[ \omega_i(1)h_{1,m}(2)^* + \omega_i(2)h_{2,m}(2)^* \right] \\
\sqrt{2} \left[ \omega_i(1)^*h_{1,m}(2) + \omega_i(2)^*h_{2,m}(2) \right]
\end{bmatrix}
\]

where Diag[.] denotes a diagonal matrix with the diagonal elements written in brackets. The interference of the User 1’s signals can be eliminated using:

\[
\frac{H_m(1)^\dagger}{|h_{1,m}|^2 + |h_{2,m}|^2} \begin{bmatrix}
r_{1,m}^* \\
r_{2,m}
\end{bmatrix} - \frac{H_1(1)^\dagger}{|h_{1,1}|^2 + |h_{2,1}|^2} \begin{bmatrix}
r_{1,1}^* \\
r_{2,1}
\end{bmatrix} = 
\]

\[
\left( \frac{H_m(1)^\dagger \Omega_m(2)}{|h_{1,m}|^2 + |h_{2,m}|^2} - \frac{H_1(1)^\dagger \Omega_1(2)}{|h_{1,1}|^2 + |h_{2,1}|^2} \right) \begin{bmatrix}
z_1(2) \\
z_2(2)
\end{bmatrix} + \begin{bmatrix}
n_1'(2) \\
n_2'(2)
\end{bmatrix}
\]

(2.41)

Similarly, the interference of User 2 can be suppressed by

\[
\frac{\Omega_m(2)^\dagger}{2 |\omega_i(1)h_{1,m}(2)^* + \omega_i(2)h_{2,m}(2)^*|^2} \begin{bmatrix}
r_{1,m}^* \\
r_{2,m}
\end{bmatrix}
\]
\[
\frac{\Omega_1(2)^\dagger}{2 | \omega_i(1) h_{1,1}(2)^* + \omega_i(2) h_{2,1}(2)^* |^2} \begin{bmatrix} r_{1,1}^* \\ r_{2,1} \end{bmatrix} = \left( \frac{\Omega_m(2)^\dagger H_m(1)}{2 | \omega_i(1) h_{1,m}(2)^* + \omega_i(2) h_{2,m}(2)^* |^2} - \frac{\Omega_1(2)^\dagger H_1(1)}{2 | \omega_i(1) h_{1,1}(2)^* + \omega_i(2) h_{2,1}(2)^* |^2} \right) \begin{bmatrix} z_1(1) \\ z_2(1) \end{bmatrix} + \begin{bmatrix} n_1'(1) \\ n_2'(1) \end{bmatrix} \tag{2.42}
\]

Note that this technique incurs noise enhancement compared to ML decoding. However, the decoding complexity of ML search can become prohibitively large. Therefore, the above interference cancellation technique makes the decoding more practical since it provides the system with single decodability. As long as the number of receive antennas is greater than or equal to the number of users in the above sub-network, we can eliminate the interference of User \( m \) on the outgoing symbols of User 1. Then, the outputs of different cancellation steps can be added to make the final equivalent channel representation of the User 1’s symbols and to decode them. A similar technique is applicable for decoding other users’ symbols. Following the elaborations in [51], the above technique can be generalized to a larger number of transmit antennas at each user in a fairly straightforward manner. However, we cannot go through more details for brevity.

## 2.4 Numerical Results

In Table 2.1, we show the number of extra complex multiplications that PODs and PSTBCs require for decoding each symbol, compared to the counterpart open-loop STBCs. According to these numbers, PODs require fewer operations at the decoder. Also in Table 2.1, the quantization computational complexities of PODs and PSTBCs are listed in the cases that we addressed in our simulations. PODs require
far fewer operations at the quantization step compared to those of PSTBCs. As a rule of thumb, the number of complex multiplications for implementing the quantizer of PSTBCs scales as $K M \bar{M}_r M_r$, whereas this number scales as $K N \bar{M}_r$ for PODs that use $N$-dimensional general vector codebooks. Note that when we use a vector codebook similar to (2.10), the number of complex multiplications at the quantizer scales as $K \bar{M}_r$.

In the first set of simulation results, in Fig. 2.3, we plot the Bit Error Rate (BER) performance of different transmission schemes that we studied in this chapter over an $8 \times 1$ MISO channel. With $K = 2$ and $K = 4$ feedback regions, the best
Figure 2.4: 4x1 MISO channel, $K = 3$ feedback regions, Rate = 1 bit/sec/Hz using BPSK symbols and Rate = 2 bits/sec/Hz using QPSK symbols.

PSTBCs are the ones that use $8 \times 4$ and $8 \times 2$ precoders, respectively, and Alamouti inner codes. The $N = 2$ and $N = 4$ PODs in these scenarios perform inferior to the corresponding PSTBCs. However, they incur lower decoding and quantization complexities. With $K = 6$, the situation is different. The $N = 6$ POD slightly outperforms the best PSTBC, the one with $8 \times 2$ precoders, although the POD’s decoding and quantization complexity is lower. Note that with $K = 8$ feedback regions, the $N = 8$ POD, the directional beamforming (BF) and the PSTBC that uses $8 \times 1$ precoders are similar structures.

In Fig. 2.4, we show the performance of the best PODs and PSTBCs over a
Figure 2.5: 4x2 MIMO channel, Rate = 1 bit/sec/Hz using BPSK symbols.

4 × 1 MISO channel, when the feedback link carries one of $K = 3$ possible feedback indices. The feedback signaling in this case can be performed for example using a 3-PSK constellation. This figure shows the simulation results for both real and complex constellations. In this case, directional beamforming provides diversity order 3. Note that here, $\frac{M_t}{K}$ is not an integer and PODs with $N = 3$-dimensional vector codebooks outperform the best PSTBCs, the ones which employ $4 \times 2$ precoder matrices. Moreover, the PODs have less complexity requirements.

The performance of the proposed design techniques for MIMO transmission is first examined over a $4 \times 2$ channel, as plotted in Fig. 2.5. Here, regardless
of the number of receive antennas, the PSTBCs with $M_t \times M$ precoders provide the best error performance when $\frac{M}{M_t} = K$. However, when $\frac{M}{K}$ is not an integer, PSTBCs can be outperformed by $N = K$ PODs. According to Table 2.1, again the complexity of implementing PODs is lower than that of PSTBCs.

Finally, the performance of the most general example of this chapter is depicted in Fig. 2.6. The code structures in this experiment are described in Section 2.2.3 and are examined over a $5 \times 2$ MIMO channel. The values of the power loading parameters reported in the example of Section 2.2.3 are optimized through this Monte Carlo simulation. Using this example, we again show that with vector
codebooks and more noticeably, with matrix codebooks, PODs can outperform PSTBCs.

Our extensive simulations, some reported above, have shown that when $\frac{M_t}{K}$ is an integer, PODs cannot outperform a PSTBC with $M = \frac{M_t}{K}$ inner STBC code rows. However, in these cases, PODs with $N = K$-dimensional beamformers are low-complexity alternatives that can provide performance results close to those of PSTBCs. When $\frac{M_t}{K}$ is not an integer, however, combining $N = K$-dimensional beamforming and coding by PODs is superior to PSTBCs. Furthermore, in these cases, the best performance results are attributed to $M_t \times T$ PODs that use
\( N = \min\{M_t, KM\} \times M = \left\lfloor \frac{M_t}{K} \right\rfloor \) precoders and add \( M_t - N \) other length \( T \) orthogonal rows. Note also that when \( \frac{M_t}{K} \leq 1 \), the best transmission strategy is directional beamforming.

The superiority of combining coding and precoding (or beamforming) by PODs to pure precoding by PSTBCs in the latter cases can be resulted from the packing properties of the precoder or beamformer codebooks with those cardinalities and dimensions \([70, 98]\). We leave further investigations of this matter for future work.

In the last experiment, we show the performance of the interference cancellation technique that we developed using PODs. In Fig. 2.7, the multi-user example in Section 2.3 is simulated and the BER performances of the two users, one with beamforming and the other in open-loop mode are shown. Using only \( M_r = 2 \) receive antennas, the diversity order 2 is provided to both users with interference cancellation. Note that using this semi-closed-loop structure, not only does the BER performance of the user with beamforming (thus the average BER of the system) outperform the open-loop code from \([51]\), but also the same single decodability feature is maintained. Based on the example of Section 2.3, one can develop other multi-user scenarios with closed-loop or open-loop users. Our primary results show that with as few receive antennas as the number of the users, a diversity order equal to the number of transmit antennas on each user can be provided to them. Increasing the number of receive antennas beyond the number of users yields multiplicative diversity order in theory. With \( M_r \) receive antennas we can obtain the diversity order \( M_t(M_r - J + 1) \) \([51]\). In Fig. 2.7, we also show the performance of the system with \( M_r = 3 \) receive antennas, which in theory provides diversity order 4.
2.5 Conclusions

In this chapter, we presented the design of PODs, a general class of matrix constellations that create a larger solution space for combining space-time coding and beamforming, compared to conventional PSTBCs. Our focus was to establish low-complexity combining schemes in the decoding and quantization stages. Through pairwise error probability analysis, and based on high-resolution quantizer assumptions, we analytically optimized POD structures with vector beamformers over MISO channels. We also showed that the derived properties are extendable to transmission across MIMO channels and to the PODs with matrix precoders. It was shown that PSTBCs are a sub-class of PODs. If $\frac{M_t}{K}$ is not an integer, PODs outperform PSTBCs.

Based on the MIMO transmission schemes studied, we also constructed PODs applicable to multiple-antenna multi-user channels, to combine beamforming and space-time coding with spatial multiplexing. For this scenario, a very low-complexity interference cancellation and decoding technique was established that scales the diversity order of the system with the number of transmit and receive antennas.
Table 2.1: Number of extra complex multiplications for quantization per frame and decoding per symbol per receive antenna.

<table>
<thead>
<tr>
<th>4x1 system</th>
<th>Quant. per frame</th>
<th>Decode per symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = 2, (M = 2)$ PSTBC</td>
<td>20</td>
<td>8</td>
</tr>
<tr>
<td>$K = 2, N = 2$ POD</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$K = 3, (M = 2)$ PSTBC</td>
<td>30</td>
<td>8</td>
</tr>
<tr>
<td>$K = 3, N = 3$ POD</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$K = 4, (M = 1)$ PSTBC</td>
<td>20</td>
<td>4</td>
</tr>
<tr>
<td>$K = 4, N = 4$ POD</td>
<td>4</td>
<td>4</td>
</tr>
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<table>
<thead>
<tr>
<th>8x1 system</th>
<th>Quant. per frame</th>
<th>Decode per symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = 2, (M = 4)$ PSTBC</td>
<td>72</td>
<td>32</td>
</tr>
<tr>
<td>$K = 2, N = 2$ POD</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$K = 4, (M = 2)$ PSTBC</td>
<td>72</td>
<td>16</td>
</tr>
<tr>
<td>$K = 4, N = 4$ POD</td>
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<td>4</td>
</tr>
<tr>
<td>$K = 6, (M = 2)$ PSTBC</td>
<td>108</td>
<td>16</td>
</tr>
<tr>
<td>$K = 6, N = 6$ POD</td>
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<td>6</td>
</tr>
<tr>
<td>$K = 8, (M = 1)$ PSTBC</td>
<td>72</td>
<td>8</td>
</tr>
<tr>
<td>$K = 8, N = 8$ POD</td>
<td>8</td>
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<th>Decode per symbol</th>
</tr>
</thead>
<tbody>
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<td>40</td>
<td>16</td>
</tr>
<tr>
<td>$K = 2, N = 2$ POD</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
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<tr>
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<td>6</td>
</tr>
<tr>
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<td>8</td>
</tr>
<tr>
<td>$K = 4, N = 4$ POD</td>
<td>16</td>
<td>8</td>
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<th>5x2 system</th>
<th>Quant. per frame</th>
<th>Decode per symbol</th>
</tr>
</thead>
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<tr>
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<td>30</td>
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We study point-to-point closed-loop communications over multiple-input single-output (MISO) quasi-static Rayleigh fading channels. This channel model characterizes the downlink of several wireless communication systems such as the new generation of fixed and mobile cellular systems in IEEE 802.16 [42]. We assume that the single-antenna receiver is able to estimate the channel coefficients perfectly. Acquiring perfect receiver channel state information (CSI) is relatively easy. Because by definition, in quasi-static channel environments, there is a long fading block length and accurate channel estimation through training is possible. Obtaining transmitter CSI (CSIT), however, requires the use of feedback in the system. When CSIT is available either fully or partially, directional beamforming can increase the average received signal-to-noise ratio (SNR). This benefit of beam-
forming is called array gain. It improves the capacity of wireless communication systems [43, 76, 85], reduces the outage probability [3, 99], and enhances the error performance. Partial CSIT is widely used in the literature for combining space-time coding and beamforming using precoded space-time codes, [42, 48, 67, 82, 104, 105], and for combining beamforming and power control [60, 63].

We design a fixed-rate practical coding and beamforming system with instantaneous power constraint at the transmitter. The performance measure is the pairwise error probability (PEP) of transmission over a finite length data frame. The diversity order in this context is defined as the decay rate of the pairwise error probability with SNR. In MISO systems, full-diversity order means an error probability decay rate equal to the number of transmit antennas. In quasi-static environments, channel fading coefficients remain constant during the transmission of each data frame, which occurs within the channel coherence time, and change independently afterwards. Therefore, feedback information must be updated in every frame. Usually, feedback channels are severely bandwidth and power limited. It is preferable to reduce the feedback rate, using resolution-constrained (quantized) CSI [70, 71, 75, 81, 102, 106]. Most of the work in the literature assume error-free (noiseless) feedback. However, this assumption requires tremendous protection of feedback information in a practical system, which requires the dedication of excessive bandwidth and power to feedback channels. We study a practically motivated scenario, where not only is the feedback link resolution constrained, but also the existence of noise in the feedback link introduces errors to feedback information [65]. This situation can naturally arise in a variety of applications, where feedback information must be carried through actual fading channels [40]. If the system is designed robustly against feedback errors, a significant portion of the ad-
ditional bandwidth and excessive power spent for protecting feedback information can be saved. This can increase the efficiency of the overall network.

For simplicity, we consider a simple discrete memoryless channel (DMC) model of the feedback link, using binary symmetric channel (BSC) models for feedback bits. We assume that the cross-over probability (bit error probability) of feedback bits is fixed over all channel realizations. This is a simple, yet practical version of the bit error model in [96], which introduces a finite-state fading channel model. A more elaborate model of feedback errors can be found in [15]. Other than the limited bandwidth and feedback errors, possible delay in feedback can also be problematic. With delay, feedback information might become outdated. Therefore, the system performance degrades due to the mismatch between the CSIT and the real channel realization [41, 60]. However, we only consider a simple erroneous feedback channel model to establish the design guidelines and leave the extensions to other feedback channel models for future work.

If feedback information is error-free, beamforming or combining space-time coding and beamforming can provide full-diversity order on top of array gain [72, 101].
However, with feedback errors, achieving full-diversity order is not straightforward [40, 65]. It was recently shown that with less restriction on the feedback rate, we can obtain full-diversity order by combining space-time coding and beamforming [1, 34, 64, 68, 72, 74]. This is even possible with feedback errors in some scenarios. For example in [49] this issue is addressed by appropriate spatial power allocation using mean and covariance feedback in block-fading channels. In the proposed combining scheme, the goal is to address the low-rate feedback and feedback error problems at the same time for quasi-static channels. Moreover, we want our scheme to extend easily for any feedback rate and feedback error probability, even if there is a mismatch between the actual feedback channel parameters and the knowledge of the transmitter/receiver sides about these parameters.

3.0.1 Organization and Notation

The rest of the paper is organized as follows: In Section 3.1, the system model is sketched and our combining scheme is introduced. This scheme is based on a code structure named generalized partly orthogonal designs (PODs). Then we derive the design criteria of our combining scheme, based on PEP analysis. In Section 3.2, using channel optimized vector quantizers (COVQs) similar to [24, 26, 61], we propose a quantizer structure for the feedback channel and a precoder structure for generalized PODs. First we consider the case when feedback error probability is known. The case of feedback channel modelling mismatch, or not knowing feedback error probability a priori is also discussed. The latter discussion is inspired from the worst-case COVQ design approaches in [45]. Section 3.3 provides the numerical results of the paper and finally Section 3.4 draws our major conclusions.

In the sequel, $p(i)$, $p(j|i)$, and $p(j;i)$ denote the marginal probability of the
event \( i \), the conditional probability of the event \( j \) given \( i \), and the joint probability of events \( j \) and \( i \), respectively. The operators \( \| \cdot \|_F \), \((\cdot)^T\), \((\cdot)^\dagger\), and \((\cdot)^*\) represent Frobenius norm, transpose, Hermitian, and conjugate, respectively. \( f(x) \) denotes the probability density function (pdf) of the random vector \( x \) and \( f(x|i) \) denotes the pdf of \( x \) given event \( i \). Finally, \( \mathbb{E}_x\{\Psi(x)\} \) shows the expectation of a function of \( x \) and \( \mathbb{E}_{x|i}\{\Psi(x)\} \) shows the same expectation given event \( i \).

### 3.1 System Components and Design Criteria

The block diagram of our system is shown in Fig. 4.1. In this system, the receiver is equipped with perfect channel estimation, ideal synchronization, and maximum-likelihood (ML) decoding.

#### 3.1.1 Forward and Feedback Channel Parameters

During the transmission of each data frame, we represent the MISO quasi-static fading forward channel coefficients by an \( M_t \)-dimensional complex Gaussian vector \( \hat{h} \). The task of the quantizer is to quantize this vector at the receiver and generate a quantization index. The feedback channel conveys the quantization index back to the transmitter. Due to the limited feedback bandwidth, in some cases we quantize a portion of \( \hat{h} \) by decomposing this vector into two parts as

\[
\hat{h} = \begin{bmatrix} \tilde{h}^T_{(M_t-N)\times 1} & \tilde{h}^T_{N\times 1} \end{bmatrix}^T
\]

(3.1)

where \( \tilde{h} \) is the \( N \)-dimensional quantized part and \( \tilde{h} \) is the \((M_t-N)\)-dimensional part which remains unknown to the transmitter. The parameter \( N \) in our system that is the dimension of the quantized vector depends on the number of feedback
indices in the feedback codebook. Note that we deal with short-term (instantaneous) power constraint at the transmitter. Also our scheme only supports fixed-rate transmissions. Therefore, the amplitude of the quantized channel vector is irrelevant to the performance and the design of our system. Our quantization scheme works on the direction of the channel sub-vector $\mathbf{h}$, i.e., $\mathbf{h} = \mathbf{H}/\|\mathbf{H}\|_F$. We decompose $\mathbf{h}$ into its amplitude and its direction through $\mathbf{h} = \sqrt{\gamma} \mathbf{h}$, where $\gamma$ is the amplitude square. The direction vector $\mathbf{h}$ is uniformly distributed over the unit amplitude complex sphere $\mathbb{C}^N$ and the amplitude square $\gamma$ is a Chi-square random variable with $2N$ degrees of freedom. The pdf of $\gamma$ is $f(\gamma) = \gamma^{N-1} \exp(-\gamma)/(N-1)!$.

In our feedback scheme, we partition the space of $\mathbf{h}$ into $K$ mutually exclusive feedback (Voronoi) regions $\mathcal{V} = \{\mathcal{V}_1, \cdots, \mathcal{V}_K\}$, whose union spans the whole unit-norm complex sphere $\mathbb{C}^N$. The objective of the quantizer at the receiver is to encode $\mathbf{h}$ by finding the index of the Voronoi region $\mathcal{V}_i$ that $\mathbf{h}$ belongs to. The objective at the transmitter is to choose a precoder matrix $\mathcal{P}_j$ based on the transmitter’s feedback index $j$, which may be different from $i$ as a result of errors in the feedback channel. We first need to design a precoder matrix codebook $\mathcal{P} = \{\mathcal{P}_1, \cdots, \mathcal{P}_K\}$ and store it at both ends of the communication link. The precoder matrix $\mathcal{P}_j$ defines a matrix constellation set at the transmitter.

In our system, after finding a feedback index, the receiver converts it to bits and sends the bits to the transmitter. The typical value of the feedback bit error probability is approximately 0.04 [40, 65]. However, for the sake of generality, we will consider a case where feedback bit error, $\rho_f$, can take any value between 0.00 and 0.50. Note that if $\rho_f > 0.50$, the transmitter can simply flip the feedback bits and achieve better performance results. In the sequel we assume that the transmitter/receiver sides of our system know $\rho_f$ a priori, unless stated otherwise.
Suppose that the feedback index $i \in \{1, \cdots, K\}$ is chosen at the receiver. The probability of this event is denoted by $p(i)$. Also, suppose that the mapping of feedback indices to feedback bits is an identity mapping. For any feedback index $i$, the feedback bits can be simply obtained as binary representation of $i - 1$. The encoder of the quantizer (and the mapper) are located at the receiver and the decoder of the quantizer (and the demapper) are implemented using a simple table look-up operation at the transmitter. In our system model, feedback indices go through a DMC from $\log_2 K$ uses of BSCs with cross-over error probabilities $\rho_f$ for each feedback bit. The feedback channel converts $i$ to $j \in \{1, \cdots, K\}$, with the conditional index inversion probability

$$p_r(j|i) = (\rho_f)^{d(i,j)} (1 - \rho_f)^{\log_2 K - d(i,j)}$$

where $d(i,j)$ is the Hamming distance of the binary representations of $j$ and $i$.

### 3.1.2 Modulation Scheme

The transmitter uses a class of coded modulation schemes called generalized partly orthogonal designs (PODs). PODs were firstly introduced in [16]. Using PODs, we can utilize feedback information from any number of channel coefficients $N$ for combining coding and beamforming across $M_t$ transmit antennas, where $M_t \geq N$. The precoder or beamformer structures incorporated in original PODs of [16] were designed based on maximizing the received SNR. The generalized PODs of this chapter are however designed based on minimizing the pairwise error probability and hence they have a different structure. Unlike the original PODs, the precoder structures used in generalized PODs allow spatial power allocation, depending on the probability of error in the feedback link. Like original PODs, generalized PODs
also have two parts: i) the STBC inner code that uses an $M_t \times T$ orthogonal design from [47, 89] and ii) the precoding part, which uses an $N \times N$ ($N$-dimensional) precoder matrix, chosen based on feedback information. Each modulation matrix in the constellation set is limited to an instantaneous power constraint $M_t T$.

For clarification, let us use the following example. Suppose that the MISO forward channel has 4 transmit antennas. Similar to [16], we construct the code structures on the columns of an orthogonal design matrix. With a slight abuse of the notation, in this example we use the following $M_t \times T$ orthogonal design structure where $M_t = T = 4$

$$
Z = \begin{bmatrix}
z_1 & -z_2 & -z_3 & -z_4 \\
z_2 & z_1 & z_4 & -z_3 \\
z_3 & -z_4 & z_1 & z_2 \\
z_4 & z_3 & -z_2 & z_1 \\
\end{bmatrix}
$$

Here, $z_\kappa : \kappa \in \{1, \cdots, T\}$ denote the data symbols chosen from a real symbol constellation. Generalized PODs allow using any precoder dimension $N$, where $N \leq M_t$. The dimension of the precoder depends on the dimension of the channel vector quantizer. For instance, using a 2-dimensional precoder, combining is performed through the following code structure:

$$
Z_j^2 = \begin{bmatrix}
z_1 & -z_2 & -z_3 & -z_4 \\
z_2 & z_1 & z_4 & -z_3 \\
\mathcal{P}_j^{2\times2} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} & \mathcal{P}_j^{2\times2} \begin{pmatrix} -z_4 \\ z_3 \end{pmatrix} & \mathcal{P}_j^{2\times2} \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} & \mathcal{P}_j^{2\times2} \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} \\
\end{bmatrix}
$$

(3.3)
Also, using a 4-dimensional precoder, we can use the following code:

$$Z^4_j = \begin{bmatrix}
P_j^{4\times4} & P_j^{4\times4} & P_j^{4\times4} & P_j^{4\times4} \\
\begin{pmatrix} z_1 \\
z_2 \\
z_3 \\
z_4 \end{pmatrix} & \begin{pmatrix} -z_2 \\
z_1 \\
-z_4 \\
z_3 \end{pmatrix} & \begin{pmatrix} -z_3 \\
z_4 \\
z_1 \\
-z_2 \end{pmatrix} & \begin{pmatrix} -z_4 \\
-z_3 \\
z_2 \\
z_1 \end{pmatrix}
\end{bmatrix}$$ (3.4)

In these codes, $P_j^{N\times N}$ denotes an $N \times N$ precoder matrix with power $N$, i.e., $\text{Tr} \left( P_j^{N\times N} P_j^{N\times N\dagger} \right) = N$, where $\text{Tr}(\cdot)$ denotes the trace operation. Note that when $N = M_t$ generalized PODs look similar to a conventional PSTBC in [72]. Also when $M_t \neq N$, they look like a POD in [16]. However, they are different structures, as we will clarify further in the sequel. Using similar approaches, one can design generalized PODs from complex orthogonal designs for any dimensions [47]. Furthermore, similar ideas can be applied to design a generalized partly quasi-orthogonal design based on quasi-orthogonal inner STBCs from [46,95]. The main challenge is how to design the precoder matrix $P_j^{N\times N}$ to obtain minimum pairwise error probability with each code structure.

Suppose that the quantizer codebook has $K$ indices. We will show that in order to obtain full-diversity order using the above codes, we must choose precoder structures with dimension $N \leq K$. If $K \geq M_t$, we have high-rate feedback and PODs with $N = M_t$-dimensional precoders can provide full-diversity. Furthermore, we will show that in an $M_t$-dimensional MISO system, where $K < M_t$, which resembles low-rate feedback in our system, the least pairwise error probability is associated to the code structures with $N = K$-dimensional precoders. As a result, to minimize PEP using a generalized POD, we should use $N = \min\{K, M_t\}$-dimensional precoders.
3.1.3 Pairwise Error Probability Analysis

Using generalized PODs, the equivalent base-band signal at the receiver can be modelled as

$$y = Z_j^\dagger \hat{h} + n$$  (3.5)

where $Z_j^\dagger$ is the transmit signal matrix (as the Hermitian of a generalized POD modulation matrix) and $n$ is the $T$-dimensional noise vector. Note that for simplicity we dropped the upper index of $Z_j$. If the regulated average SNR at the receiver is $\eta_0$, each element of $n$ is a complex circularly symmetric additive white Gaussian noise (AWGN) variable with variance $\sigma_n^2 = 1/(M_t\eta_0)$.

To decode the matrix $Z_j$, we assume that the index of the constellation set, $j$, which represents the modulation scheme at the transmitter is known at the receiver. Usually, in wireless communications standards, the transmitter sends some control signals to the users (receivers) in the header field of each data frame that indicate the modulation scheme. For example, in IEEE 802.16, downlink burst profile of the physical layer, which is a part of downlink channel descriptor of the MAC layer contains the type of modulation used [42]. Our system requires to include the index of the constellation set, $j$, in this category and to assume that the receiver knowledge of $j$ is updated as frequently as feedback is applied. By this assumption, the conditional PEP of detection in favor of an erroneous codeword $Z_j'$ when $Z_j$ is transmitted can be tightly upper bounded by the following Chernoff bound [47]

$$p(Z_j \rightarrow Z_j'|\hat{h}) \leq \frac{1}{2} \exp \left( -\frac{D(Z_j, Z_j')}{4\sigma_n^2} \right)$$  (3.6)

where $D(Z_j, Z_j') = \hat{h}^\dagger (Z_j - Z_j')(Z_j - Z_j')^\dagger \hat{h}$. Now, suppose that the receiver processes $\hat{h}$ and chooses $V_i$ from the set of Voronoi regions $\mathcal{V}$. The average probability
of pairwise error given the feedback index $i$ at the receiver can be expressed as

$$p(Z_j \rightarrow Z'_j \mid i) = \int_{\text{Dom}(\tilde{h}|i)} p(Z_j \rightarrow Z'_j \mid \tilde{h}) f(\tilde{h} \mid i) \, d\tilde{h} \quad (3.7)$$

where $\text{Dom}(\tilde{h}|i)$ is the domain of the random variable $\tilde{h}$, conditioned on the receiver’s feedback index. The variable $\tilde{h}$ can be expressed as a one-to-one function of the three variables, $\tilde{h}$, $\bar{h}$, and $\gamma$. Therefore, $f(\tilde{h} \mid i)d\tilde{h}$ is statistically equivalent to $f(\tilde{h}; \gamma; \bar{h}) \, d\gamma \, d\bar{h}$ [79]. Also, for independent and identically distributed (i.i.d.) channel realizations, the latter three variables are independent. Therefore, $f(\tilde{h}; \gamma; \bar{h} \mid i) = f(\tilde{h}) f(\gamma) f(\bar{h} \mid i)$ and we can reexpress (3.7) as

$$p(Z_j \rightarrow Z'_j \mid i) = \int_{\text{Dom}(\tilde{h})} \int_{\mathcal{R}^+} \int_{\mathcal{V}_i} p(Z_j \rightarrow Z'_j \mid \tilde{h}, \bar{h}, \gamma) f(\tilde{h}) f(\gamma) f(\bar{h} \mid i) \, d\tilde{h} \, d\gamma \, d\bar{h} \quad (3.8)$$

Here, $\mathcal{R}^+ = [0, \infty)$ is the domain of $\gamma$ and $\text{Dom}(\tilde{h})$ is the domain of the unquantized portion of the channel vector. For generalized POD structures, using Equation (3.6), the above PEP expression can be upper bounded by:

$$p(Z_j \rightarrow Z'_j \mid i) \leq \frac{1}{2} \int_{\text{Dom}(\tilde{h})} \int_{\mathcal{R}^+} \int_{\mathcal{V}_i} \tilde{h} \, d\gamma \, d\bar{h} \, f(\tilde{h}) f(\gamma) f(\bar{h} \mid i) \exp\left(-\eta_c [\gamma \beta + \vartheta]\right) \quad (3.9)$$

where $\beta = \bar{h}^\dagger \mathcal{P}_j^\dagger \mathcal{P}_j \bar{h}$ and $\eta_c = \left(\sum_{\kappa=1}^T |z_\kappa - z'_\kappa|^2\right) / 4\sigma_n^2$. Note that $\eta_c$ is proportional to the Euclidian distance of the inner code matrices and the SNR. Also $\vartheta = \|\tilde{h}\|^2_F$ takes values on $[0, \infty)$ and follows a Chi-square distribution. In the sequel, with a slight abuse of the notation, we denote $\mathcal{P}_j^{N \times N}$ by $\mathcal{P}_j$. Because, the precoder dimension $N$ is fixed and the transmitter must pick the precoder matrix from $\mathcal{P} = \{\mathcal{P}_1, \cdots, \mathcal{P}_K\}$ using feedback index $j$. To proceed, we use the following relations:

$$\int_0^\infty f(\vartheta) \exp\left(-\eta_c \vartheta\right) d\vartheta = (1 + \eta_c)^{-(M_t-N)}$$
and
\[
\int_{\mathbb{R}^+} \exp \left( -\eta_c \gamma \beta \right) I(\gamma) d\gamma = (1 + \eta_c \beta)^{-N}
\]
Then the PEP of the worst-case error event, conditioned on the receiver index \(i\) is
\[
p(Z_j \rightarrow Z'_j \mid i) \leq \frac{1}{2} (1 + \eta_c)^{-\left( M_t - N \right)} \mathbb{E}_{\tilde{H} \in \mathcal{V}_i} \left\{ \left( 1 + \eta_c \tilde{H} \tilde{P}_{j} \tilde{P}_{j}^\dagger \tilde{H} \right)^{-N} \right\}
\]
where \(\eta_c\) is the minimum Euclidian distance between the inner STBC parts of \(Z_j\) and \(Z'_j\) divided by the noise power. Note that \(\eta_c\) represents the transmission signal power divided by the noise power at the receiver antenna or the transmit SNR. When we use unit-norm symbols such as BPSK, in a worst-case error event we have \(\eta_c = M_t \eta_0/(4T)\).

3.1.4 Expanding PEP and Deriving the Design Criterion

Our goal is to minimize the average PEP over all feedback realizations:
\[
P_e = \sum_{i=1}^{K} \sum_{j=1}^{K} p(Z_j \rightarrow Z'_j ; j \mid i) \tag{3.11}
\]
\[
= \sum_{i=1}^{K} \sum_{j=1}^{K} p(j \mid i) p(Z_j \rightarrow Z'_j \mid j \mid i) \tag{3.12}
\]
\[
= \sum_{i=1}^{K} \sum_{j=1}^{K} p_t(j \mid i) p(i) p(Z_j \rightarrow Z'_j \mid i) \tag{3.13}
\]
In the above formulas, the first equality shows the summation of the probabilities that index \(i\) is transmitted over the feedback channel, index \(j\) is received at the transmitter, and a pairwise error occurs. The second equality is the application of the Bayes’ rule. Finally, the third one can be written noting that any two modulation matrices \(Z_j\) and \(Z'_j\) are in the constellation set \(j\), chosen given index \(j\) at the transmitter. Hence, the event \(Z_j \rightarrow Z'_j \mid i\) is conditioned on \(j\).
According to the average PEP expression in (3.13), the minimum PEP precoder set \( \{ P_1, \cdots, P_K \} \) can be obtained by solving the following optimization problem:

\[
\min \sum_{j=1}^{K} \sum_{i=1}^{K} p_r(j|i)p(i) \mathbb{E}_{\tilde{h} \in V_i} \left\{ \left( 1 + \eta_c \tilde{h}^\dagger P_j P_j^\dagger \tilde{h} \right)^{-N} \right\}
\]

s.t. \( \forall j, \ P_j P_j^\dagger \succ 0 \) and \( \text{Tr}(P_j P_j^\dagger) = N \) (3.14)

Here, \( P_j P_j^\dagger \succ 0 \) shows that the Hermitian matrix \( P_j P_j^\dagger \) is positive semi-definite.

We have also omitted the constant term \((1 + \eta_c)^{-(M_t-N)}\) from the objective function. Equation (3.14) implies that deriving the set of precoder matrices depends on the feedback indices at the input/output of the quantizer, the set of Voronoi regions \( V \), and the conditional index inversion probability of the feedback channel, \( p_r(j|i) \).

### 3.2 Quantizer and Precoder Codebook Design

COVQs have been proposed to minimize the quantization average distortion when the quantization indices go through erroneous channels [24,26,61]. We use this idea to design a CSI quantizer for our system. Our COVQ design problem is to find the pair of Voronoi regions and precoder matrices \( (V, P) \) that solves the optimization problem expressed in (3.14).

First, note that given a fixed set of Voronoi regions \( V \), the joint optimization in (3.14) can be decoupled into a series of individual optimizations for each \( P_j \) as

\[
\min \sum_{i=1}^{K} p_r(j|i)p(i) \mathbb{E}_{\tilde{h} \in V_i} \left\{ \left( 1 + \eta_c \tilde{h}^\dagger P_j P_j^\dagger \tilde{h} \right)^{-N} \right\}
\]

s.t. \( P_j P_j^\dagger \succ 0 \) and \( \text{Tr}(P_j P_j^\dagger) = N \) (3.15)

This property simplifies the design procedure significantly. In (3.15), both the objective function and the constraints are convex functions of \( P_j \). Furthermore, the objective function is differentiable throughout the whole domain of \( P_j \). Therefore,
we can use a steepest descent type of algorithm, such as the gradient algorithm to solve this problem [78]. We successively find the Voronoi regions associated to each index \(i\) from (3.14) and the precoder matrices from separate implementations of (3.15) for every \(j\). This algorithm proceeds as follows:

### 3.2.1 Training-Based Gradient Algorithm:

1) Generate a sequence of training samples of \(\vec{h}\) by normalizing a sequence of \(N\)-dimensional complex Gaussian random vectors. Then assume an initial set of positive semi-definite precoder matrices \(\mathcal{P} = \{\mathcal{P}_1, \cdots, \mathcal{P}_K\}\) with powers \(\|\mathcal{P}_j\|_F^2 = N, \forall j\).

2) Assign index \(i\) to each training vector \(\vec{h}\) if

\[
i = \arg \min_{\iota \in \{1, \cdots, K\}} \sum_{j=1}^{K} p_t(j|\iota) \left(1 + \eta_c \vec{h}^\dagger \mathcal{P}_j \mathcal{P}_j^\dagger \vec{h}\right)^{-N}
\]

The set of training vectors with assigned index \(i\) statistically represent \(\mathcal{V}_i\). Note that the above formula will be used later in the encoder of the quantizer after the codebook \(\mathcal{P}\) is designed.

3) To optimize the precoder matrix from (3.15), first define the following objective function for each index \(j\):

\[
J(\mathcal{P}_j) = \sum_{i=1}^{K} p_t(j|i)p(i) \mathbb{E}_{\vec{h}\in\mathcal{V}_i} \left\{ \left(1 + \eta_c \vec{h}^\dagger \mathcal{P}_j \mathcal{P}_j^\dagger \vec{h}\right)^{-N} \right\}
\]

The gradient of \(J(\mathcal{P}_j)\) can be expressed as

\[
\nabla J(\mathcal{P}_j) = -2N\eta_c \sum_{i=1}^{K} p_t(j|i)p(i) \mathbb{E}_{\vec{h}\in\mathcal{V}_i} \left\{ \left(1 + \eta_c \vec{h}^\dagger \mathcal{P}_j \mathcal{P}_j^\dagger \vec{h}\right)^{-N-1} \vec{h} \vec{h}^\dagger \mathcal{P}_j \right\}
\]

In numerical implementation of (3.18), note that the random vector \(\vec{h}\) is uniformly distributed on the region \(\mathcal{V}_i\) and its pdf is proportional to the volume of \(\mathcal{V}_i\), or
the marginal probability $p(i)$. For any function $\Psi$, to find $p(i) \mathbb{E}_{\tilde{h} \in V_i} \{\Psi(\tilde{h})\}$, it is sufficient to add the values of $\Psi(\tilde{h})$ throughout the partition of the training space that is represented by index $i$ (approximation of $V_i$) and divide the result by the size of the training sequence.

4) To proceed with the gradient algorithm, update each matrix in the precoder codebook $\mathcal{P}$ using the following relation:

$$\mathcal{P}_j(t + 1) = \left[ \mathcal{P}_j(t) - \alpha(t) \nabla J(\mathcal{P}_j(t)) \right]_N^+$$  \hspace{1cm} (3.19)

In the above formula, $\mathcal{P}_j(t)$ denotes the value of the precoder matrix $\mathcal{P}_j$ in the $t$-th iteration of the algorithm. The step size $\alpha(t)$ can be set to any decreasing function of $t$, but from [78] we use $(1 + m)/(1 + t)$ with an arbitrary positive real number $m$. Furthermore, the operation $[\mathcal{X}]_N^+$ shows that $\mathcal{X}\mathcal{X}^\dagger$ is projected onto the space of positive semi-definite matrices with power $N$. We use the Euclidean distance as the projection measure. With this measure, the above projection is simply removing all the negative eigenvalues of the matrix inside the brackets and normalizing the matrix, so that its power is $N$ [7].

5) Use the output of the above algorithm and improve it by successively implementing steps (2) and (3-4) until convergence. Since the resulting sequence of objective function values from (3.14) is decreasing and PEP is bounded below, the convergence of this algorithm is guaranteed. However, we cannot claim that by the above alternation we converge to a globally optimal solution.

After storing the precoder matrix codebook $\mathcal{P}$ at the transmitter/receiver ends of the system, the encoder of the quantizer at the receiver side operates similar to Equation (3.16). At the transmitter side, the task of the quantizer decoder is to choose $\mathcal{P}_j$ from the precoder codebook.

In general, we can implement different index mapping methods in our scheme.
For incorporating a mapping other than identity, we map the output index of the quantizer $i$ to $i'$ at the receiver side and pass it through the feedback channel. Then at the transmitter side, we use inverse mapping (demapping) to obtain the quantization index $j$ from the output of the feedback channel, i.e., $j'$. Note that to drive the system with different mappings, one should replace $p_f(j|i)$ with $p_f(j'|i')$ in the design of the precoder codebook and in the implementation of the quantizer’s encoder.

### 3.2.2 System Design Without Knowing $\rho_f$ A Priori

The design techniques established in the previous section can be extended to the case that the knowledge of $\rho_f$ is not accurate. Suppose that we know a coarse range of feedback bit error probability $\rho_f$ at the transmitter/receiver sides. The uncertainty about the feedback channel model can be taken into account assuming that $f_a \leq \rho_f \leq f_b$, where $f_a, f_b \in [0, 0.5]$. This type of channel modelling mismatch results from the uncertainty about the feedback channel conditions in a wireless environment and the uncertainty about the amount of resources that different receivers in the network may spend for protecting feedback signals.

We develop our design strategy based on assuming a fixed $\rho_d \in [f_a, f_b]$, which is called the design cross-over probability parameter. Inspired from [45], it is known that by designing COVQs based on worst-case assumptions, the system enjoys desirable robustness against channel modelling mismatch. Therefore, we find the set of precoder matrices and Voronoi regions that minimize the worst possible pairwise error probability. Similar to (3.14), the precoder design criterion can be
expressed as:

$$\min \max \sum_{j=1}^{K} \sum_{i=1}^{K} p_d(j|i)p(i) \mathbb{E}_{\eta \in \mathcal{V}_i} \left\{ \left( 1 + \eta c^\dagger P_j P_j^\dagger \right)^{-N} \right\}$$

$$\rho_d \in [f_a, f_b] \quad \text{s.t.} \; \forall \; j, \; P_j P_j^\dagger > 0 \; \text{and} \; \text{Tr}(P_j P_j^\dagger) = N$$

(3.20)

where $p_d(j|i)$ is the conditional index inversion probability between $i$ and $j$ assuming that the BSC cross-over probability is $\rho_d$. It is straightforward to show that the objective function in (3.20) is an increasing function of the design cross-over probability parameter $\rho_d$. We can show this property by plotting $P_e$ from (3.13) versus the SNR. Therefore, the min max value of $\rho_d$ coincides with the maximum point of the cross-over probability range, which is $\rho_d = \max \rho_f = f_b$. By this choice, the rest of the COVQ design procedure from Section 3.2.1 can be readily applied.

If besides the boundaries of the error range, the feedback error distribution is also known a priori, the optimal design method is to use the average conditional index transition probabilities, $\overline{p_d}(j|i)$ in the COVQ design procedure [45]. For example, if $\rho_f$ is uniformly distributed over $[f_a, f_b]$, the average conditional probabilities can be expressed as

$$\overline{p_d}(j|i) = \frac{1}{f_b - f_a} \sum_{\kappa=0}^{r-d_{i,j}} \binom{r-d_{i,j}}{\kappa} \left( -1 \right)^{r-d_{i,j} - \kappa} \frac{(f_b)^{r-\kappa+1} - (f_a)^{r-\kappa+1}}{r-\kappa+1}$$

(3.21)

which can be used in designing the COVQ and the precoder matrices. In the numerical results of the next section, we study different mismatch cases and illustrate the system performance under the aforementioned conditions.
3.3 Numerical Results

3.3.1 Precoder Characteristics

In this section, we study the precoder structure of a 4-antenna system \((M_t = 4)\) with 16 feedback indices \((K = 16)\) or 4 feedback bits \((\log_2 K = 4)\). The precoder codebook is designed for different cross-over probabilities \(\rho_f\). The solutions are obtained by solving optimization problem (3.14), using the training based gradient algorithm explained in Section 3.2.1. Fig. 3.2 depicts the eigenvalues of the first member of the precoder codebook, \(P_1\), i.e., \([\delta_1, \cdots, \delta_4]\). Other members of the codebook also have similar properties. Here, \(P_1 P_1^\dagger = U_1 \text{Diag}[\delta_1^2, \cdots, \delta_4^2] U_1^\dagger\), where \(U_1\) is unitary and \(\text{Diag}[\cdot]\) denotes a diagonal matrix. Note that the trans-
mitter’s power constraint requires that $\sum_{\kappa=1}^{4} \delta_{\kappa}^2 = 4$. When the feedback link is error-free or $\rho_f = 0$, all the transmission power is assigned to the first eigenmode, denoted by the largest eigenvalue, $\delta_1$. In other words, to achieve minimum PEP, transmission power is projected onto the direction of the major eigenvector of the precoder matrix, i.e. a column of $U_1$. This result coincides with the optimality of directional beamforming. With an error-free feedback link, we can design a directional beamforming system, where the transmit signals are the product of scalar symbols and unit-norm beamforming vectors. Hence, our precoding system degenerates to a beamforming system, where the first eigenvector of the precoder matrix is equivalent to the beamforming direction. As the cross-over probability of the feedback link increases, the eigenvalues of the minimal PEP precoder matrix converge to equal values. A better error performance can be obtained by spreading the power among different directions in space. In this case, our precoded matrix constellations also converge to matrices with equal eigenvalue squares, similar to open-loop orthogonal space-time block codes.

3.3.2 BER Computations

Fig. 3.3 shows the Bit Error Rate (BER) of our combining scheme over a $4 \times 1$ MISO channel using Monte Carlo simulations. The feedback link carries 4 bits with different cross-over probabilities. This situation resembles a high-rate feedback assumption $K \geq M_t$. The transmitter uses BPSK constellation symbols and the transmission rate is 1 bit/sec/Hz. In each data frame we transmit 130 data symbols. In this figure, we demonstrate the performance of an open-loop STBC and that of the combining scheme without feedback errors. By increasing the cross-over probability $\rho_f$ from 0.00 to 0.50, the BER of the system ranges between
Figure 3.3: Bit Error Rate for 4-ant. generalized PODs with 4 bits per vector COVQs. Rate = 1 bit/sec/Hz using BPSK symbols.

a directional beamforming system and an open-loop system. All the curves show full-diversity order within the range of SNRs that we consider in our simulations. This figure shows that even with feedback errors, our combining scheme preserves full-diversity order and it also provides additional array gain compared to an open-loop STBC. In this figure, we also plot the BER of Grassmannian beamforming from [70,72] and show that it is optimal in terms of minimizing the PEP when feedback is error-free. According to our numerical experiments, different index mapping/demapping schemes at the input/output of the feedback channel result in similar performance results if the cross-over probability of the feedback link is
Figure 3.4: Bit Error Rate for 6-ant. generalized PODs with 2 bits per vector COVQs. Rate = 1 bit/sec/Hz using BPSK symbols.

known and the precoder codebook is optimized for each specific mapping.

The BER performance of our scheme with low-rate and also erroneous feedback is illustrated in Fig. 3.4. In this experiment, we consider a $6 \times 1$ MISO channel with 2 feedback bits or 4 feedback regions. Directional beamforming in this case cannot achieve full-diversity order. Our combining system also shows the same property if we use 6-dimensional precoder matrices. In order to obtain full-diversity order, PSTBCs in the literature for instance the unitary PSTBCs in [72] use $6 \times 2$ precoder matrices from unitary constellations in [38], applied to $2 \times T$ orthogonal design matrices [47]. We use generalized POD constellation matrices similar to the one introduced in (3.3), based on a $6 \times 8$ orthogonal code structure. This code is
Figure 3.5: Bit Error Rate for 4-ant. generalized PQODs with 2 bits and 1 bit per vector COVQs. Rate = 2 bits/sec/Hz using QPSK symbol constellations with $\pi/4$ rotations.

generated by removing two rows of an $8 \times 8$ orthogonal design, leaving the first two rows without precoding, and multiplying $4 \times 4$ precoder matrices inside the code structures at the lower 4 rows of the matrix. The simulation curves can be extended using other values of $N$, which we skip for the sake of clarity in the figure. Our conclusion from this study is that only when $N \leq K$, full diversity order can be obtained. Moreover, when $K < M_t$, minimum BER performance is attributed to the POD structure with $K$-dimensional precoders. Over an error-free feedback channel, our combining scheme with 4-dimensional precoders outperforms PSTBCs with $6 \times 2$ unitary precoders. Fig. 3.4 also shows the performance
of the system with different feedback cross-over probabilities. With a non-zero cross-over probability in the feedback link, PSTBCs with unitary precoders fail to obtain full-diversity order. Our combining scheme with 4-dimensional precoder matrices, designed based on knowing $\rho_f$, however preserves full-diversity order, even if feedback is low-rate and erroneous.

In Fig. 3.5, we show the performance of our combining schemes over a 4-antenna MISO channel with complex constellations. There is no $M_t \times T$ rate-one complex orthogonal design for $M_t > 2$ [47]. In this case, instead of using orthogonal design structures with less rates, we can use a quasi-orthogonal space-time block code (QOSTBC) [46,95], to obtain a generalized partly quasi-orthogonal design (PQOD) structure. The quasi-orthogonal inner code of this example uses QPSK modulation symbols with $\frac{\pi}{4}$ rotations. The transmission rate is 2 bits/sec/Hz. With $K = 4$ and $K = 2$ feedback regions, we use generalized PQODs with $N = 4$ and $N = 2$-dimensional precoders, respectively. Again, with feedback errors, these constellations can provide full-diversity order and additional array gain compared to open-loop QOSTBCs. In this figure, we also demonstrate the performance of this system employing Grassmannian beamforming, when feedback is erroneous. This situation resembles a noiseless design mismatch and results in not achieving full-diversity order. Here, the number of feedback regions is equal to the number of transmit antennas and Grassmannian beamforming is similar to antenna selection (AS). The performance of our combining scheme designed assuming noiseless feedback (VQ-design) is also very close to that of Grassmannian beamforming. In this case, there are $N$ independent beamforming directions in space with similar cross (Chordal) distances [70,72]. Upon receiving an erroneous feedback index, it does not matter which direction we pick. Hence, employing different index mapping
schemes in this case does not change the performance of the system, even when
the feedback channel is erroneous.

3.3.3 Effect of Mismatch

In the last experiment, we investigate the effects of channel modelling mismatch
on the performance of our system. For the numerical results shown in Fig. 3.6, we
again consider a $4 \times 1$ MISO channel with 16 feedback regions and BPSK symbol
constellations. We assume certain feedback cross-over probabilities for designing
the COVQ or VQ-based combining system and use different ones for Monte Carlo
simulations. The following cases are studied:

First, the precoder codebook is designed assuming feedback is noiseless. We
call this design a VQ-based one since COVQ becomes a VQ when $\rho_d = 0.00$. Also
the combining scheme degenerates to directional beamforming. The resulting sys-
tem is examined using a feedback link, where the actual cross-over probability is
$\rho_f = 0.04$. The performance results are very close to the performance of Grass-
mannian beamforming when operating over an erroneous feedback channel with
$\rho_f = 0.04$. This type of mismatch, which results in severe performance degrada-
tion is referred to as VQ-based design mismatch. In this case, since the number of
feedback regions is more than the number of transmit antennas, different beam-
forming directions have different cross (Chordal) distances [70,72]. Hence, we can
improve the performance of the system by modifying the feedback index mapping
without any additional cost. To show this property, we first define the average
Chordal distance of the beamforming directions at the transmitter, conditioned
on the feedback indices at the receiver, when the feedback cross-over probability
is given. Then we minimize this average distance over the space of possible index
mapping solutions using simulated annealing (SA) [25]. The details of this algorithm are skipped for brevity. The resulting VQ-based system outperforms the same system with identity mapping. However, the gain of optimizing the index mapping is not significant compared to the gain of COVQ-based design since it does not change the diversity of the system. Second, we assume the design cross-over probability $\rho_d = 0.04$ and examine the system performance over an error-free feedback link, where $\rho_f = 0.00$. This situation is called COVQ-based design mismatch. This type of mismatch does not degrade the performance of the system severely. The performance results in this case are worse than those of a system with ideal design of precoders ($\rho_d = 0.00$) and error-free simulations ($\rho_f = 0.00$). However, full-diversity order is still preserved.

Third, we consider a feedback channel where the cross-over probability changes uniformly in the range $0.00 \leq \rho_f \leq 0.04$. In this scenario, as we explained in Section 3.2.2, we use a worst-case design strategy to obtain robustness against feedback channel modelling mismatch. The precoder codebook is designed assuming that the cross-over probability is $\rho_d = 0.04$. By this choice, full-diversity order is preserved and also the array-gain of the system is superior to an open-loop system. Fourth, we consider the same feedback channel model with the same range of cross-over probabilities, but this time we assume an alternative design parameter, where $\rho_d$ is the average value of feedback errors, i.e., $\rho_d = (f_a + f_b)/2 = 0.02$. By this assumption, the transmission scheme does not achieve full-diversity order. The latter observation confirms that the worst case design assumption $\rho_d = \max[f_a,f_b]$ is a better candidate compared to the average error design. Fifth and finally, we assume that we know the error distribution a priori. Then the optimal design method is to use the average conditional index transition probabilities, $p_d(i|j)$ in
Figure 3.6: Bit Error Rate for 4-ant. generalized PODs with 4 bits per vector COVQs. Rate = 1 bit/sec/Hz using BPSK symbols with feedback channel modelling mismatch, ($\rho_d \neq \rho_f$).

the design procedure as proven in [45], where the average probabilities can be calculated by (3.21). The performance of the system with this design method is depicted in Fig. 3.6, which shows that knowing the error distribution can notably improve the array gain of the system compared to the worst-case design.
3.4 Conclusions

In this chapter, we studied high-rate and low-rate feedback (closed-loop) communication systems with possible feedback errors. We showed numerically that our combining schemes obtain full-diversity order with additional array gain compared to open-loop systems. The design criterion of our scheme was derived based on a pairwise error probability measure and our system was optimized using a training-based algorithm. As the feedback cross-over probability approaches zero, our scheme converges to a directional beamforming system. On the other hand, as the feedback error increases, our scheme converges to a no-CSIT or open-loop system. We presented combining strategies for both high-rate and very low-rate feedback scenarios, which is simply extendable to the cases that the feedback link is noisy. With very low-rate feedback, the dimension of the precoder matrix in our scheme reduces and our system converges to an open-loop STBC structure.

The design strategies and solution algorithms presented in this chapter are robust against feedback channel modelling mismatch. Using a worst-case design approach, we designed the quantizer structure and derived the precoder matrices used in the code structures, even if the exact value of the feedback channel error is unknown to the transmitter/receiver sides a priori. It was shown numerically that even in this case, our transmission scheme preserves full-diversity order and provides additional array gain compared to open-loop systems.
Chapter 4

Outage Behavior of Slow Fading Channels with Power Control Using Partial and Noisy CSIT

For delay-unconstrained transmission over wireless fading channels, transmission data codewords can span over an infinite number of independent fading realizations. Consequently, ergodic capacity is a valid performance measure [30]. In practice, however, delay requirements may limit the transmission to a finite number of fading blocks. Therefore, block fading channel models are more appropriate [57]. In this context, we are interested in very slow fading scenarios where each codeword is transmitted over a single fading block. This channel model is also called quasi-static in the communications literature. In this case, the conventional performance measure is the outage probability [6,8]. Over the duration of one codeword, the channel coefficients remain constant. Acquiring perfect receiver channel state information (CSI) in this case is relatively simple. Because
by definition, we have a long fading block length and accurate channel estimation through training is possible. Obtaining transmitter CSI (CSIT), however, requires feedback in the system. The optimal outage probability of this system with perfect feedback is studied thoroughly in the literature and it is known that power control at the transmitter significantly reduces the outage probability [6, 8].

The more practical scenario where the feedback information is resolution constrained or rate-limited is also studied in the literature using power control approaches [53–56, 58, 59]. In this case, to enable power control in the system, a quantizer is defined at the receiver/transmitter ends, so that the receiver can convey an index from a quantization codebook back to the transmitter that represents the state of the channel. Kim et al. proposed a quantizer structure for power control in [53, 56] and pointed out that both “strongest” and “weakest” channel realizations must be represented by the same index. In other words, the quantizer structure is circular. Moreover, they proved that with error-free feedback, the optimal quantizer structure has contiguous Voronoi regions.

Quantized feedback is also extensively used in the literature for beamforming over multiple-antenna channels and elegant beamforming strategies such as Grassmannian beamforming have been introduced [70, 75]. These beamforming schemes do not require any power adaptation at the transmitter, however the maximum diversity gain that they can provide is equal to the diversity gain of a no-CSIT system. Note that with the outage probability performance measure, the diversity gain is defined as the decay rate of the outage probability with respect to the signal-to-noise ratio (SNR).

Here, we study a more practical scenario, where not only is the feedback link resolution constrained, but also the feedback indices are subject to errors. This sit-
Evaluation naturally arises in a variety of applications since feedback information must also be carried through a fading channel with limited power resources. Moreover, communicating feedback information is severely time-sensitive, and the number of feedback channel uses must be limited. From a practical point of view, we need to minimize the delay between the channel estimation and quantization instance at the receiver and the feedback signal detection instance at the transmitter. Because the transmitter can only adapt the transmission codeword power level, upon receiving (decoding) the feedback index and within the time of communicating feedback, it should remain idle. These limitations of the feedback stage of a closed-loop communication system justifies a noisy channel model for the feedback link.

For simplicity, we consider a discrete memoryless channel (DMC) model of the feedback link, where the bit error (cross-over) probabilities are constant for all channel realizations. A more elaborate finite-state feedback channel model can be found in [15] in a different context. Since we consider slow fading channels, we
assume that the channel remains constant during the transmission of $N$ consecutive data symbols. Moreover, we assume that $N$ is large enough so that a rate equal to the instantaneous mutual information of the channel is achievable. We won’t consider any feedback delay in our study. We will assume that while the feedback information reaches the transmitter, the channel remains constant, so it does during transmission of one codeword of $N$ channel uses.

Throughout this chapter, we also assume that the transmission rate is constant and is known to the receiver. Therefore, regardless of the noisy feedback index appearing at the transmitter, the receiver attempts to decode the received codeword from a codebook of known rate. If the transmission power is greater than or equal to the power needed for successful (outage-free) transmission or in other words, channel inversion, then the decoding succeeds. Otherwise outage occurs. This is in contrast to the assumptions made in rate-adaptive strategies, where the receiver needs to know the feedback index that appears at the transmitter [15,20].

To design the optimal CSI quantizer for this system, we solve an outage minimization problem under an average (long-term) power constraint at the transmitter. Interestingly, the structure of our quantizer becomes similar to a channel optimized scalar quantizer (COSQ) [24–26,61]. As the quality of the feedback link degrades, the reconstruction points of the power control codebook merge and the outage performance of the system approaches the no-CSIT performance [103].

It has been also shown that the diversity gain of a system with power control based on quantized feedback increases polynomially with the cardinality of the power control codebook if the feedback channel is error-free [53–56,59]. This result motivates us to study the diversity gain of our transmission scheme, where the feedback information is erroneous. Through asymptotic analysis, we show
that the achievable diversity gain of the system is equal to the diversity gain of a no-CSIT scheme. Note, however, that by proper design of the quantizer, we can achieve less outage probabilities compared to a no-CSIT scheme.

4.0.1 Organization and Notation

The rest of the paper is organized as follows. In Section 4.1, we introduce the forward link transmission and the feedback channel model. Numerical design of the quantizer is proposed in Section 4.2 and the optimality of the proposed quantizer structure is proven in Section 4.3. In Section 4.4, we present the asymptotic analysis of the outage probability with noisy feedback and derive the diversity gain of the system. In Section 4.5, the outage performance of the system is numerically evaluated. Finally, we draw the major conclusions of the work in Section 4.6. Also, the proofs of the lemmas and the theorems are provided in Section 4.7.

In the following sections, we use \( P \) for power-related variables and \( P \) or \( p \) to represent probabilities. Furthermore, all the logarithms are natural unless otherwise stated. For two sets \( A \) and \( B \), \( A - B \) represents the set of the members of \( A \) that do not intersect with the set \( B \).

4.1 System Model

4.1.1 Forward Link Transmission Model

The block diagram of the system under study is shown in Fig. 4.1. In this scheme, the \( t \)-antenna transmitter scales a rate \( R \) nats per channel use \( t \)-dimensional complex Gaussian codeword \( C \) by the CSIT-dependent power loading factor \( \sqrt{P_i/t} \). This power control parameter is dictated by the receiver via the feedback index \( i \).
The covariance matrix of the transmission codeword is \( \mathbb{E}\{CC^\dagger\} = I_t \), the identity matrix, where \( \mathbb{E}\{\cdot\} \) denotes ensemble expectation and \( \dagger \) represents the complex conjugate transpose operation.

The received signal in each channel codeword can be represented by

\[
y(l) = H x(l) + n(l) \quad \forall l \in \{1, \cdots, N\}
\]

where \( l \) denotes the time index within a block of length \( N \) channel uses. In this model, \( n(l) \) is a unit-power circularly symmetric complex Gaussian noise process and \( H \) is the \( r \times t \) complex Gaussian channel matrix with entries of variances 0.5 per real dimension. Furthermore, \( r \) is the number of the receive antennas. The channel coefficients remain constant over the duration of one fading block. Assume that the receiver is capable of estimating the channel matrix perfectly and there is a feedback link carrying a limited number of feedback bits from the receiver to the transmitter.

Using a scaled identity covariance codeword with power \( P \), the mutual information between the transmitter and the receiver data can be expressed as

\[
I(P) \triangleq \min\{r,t\} \sum_{\kappa=1} \log \left(1 + \frac{\lambda_{\kappa} P}{t}\right)
\]

where \( \{\lambda_{\kappa}\}, k = 1, 2, \cdots, \min\{r,t\} \) denote the eigenvalues of \( HH^\dagger \) [53]. For future references, let us define \( P_R(H) \) to be the solution of

\[
I(P) - R = 0
\]

In other words, \( P_R(H) \) is the minimum power level required at the transmitter to invert the channel or the required transmission power so that the receiver can successfully decode the codeword of rate \( R \) nats per channel use.
4.1.2 Quantizer Structure and Feedback Channel Model

The goal of the feedback link is to enable power control in the system. Suppose that the transmitter and the receiver employ a finite-sized power control codebook \( \{P_0, \ldots, P_{K-1}\} \). The receiver chooses a member of this codebook based on the realization of the channel and conveys its index back to the transmitter using \( \lceil \log_2 K \rceil \) feedback bits, where \( \lceil \cdot \rceil \) shows the smallest integer greater than or equal to a real number. Inspired by the unlimited feedback results of [8], Kim et al. proposed the following feedback index mapping to minimize the probability of outage at rate R, when the feedback link is error-free [54, 56]:

\[
J(H) = \begin{cases} 
  j & \text{if } P_R(H) \in (P_{j-1}, P_j] \\
  0 & \text{if } P_R(H) > P_{K-1}
\end{cases}
\] (4.4)

Here, for notational convenience, \( P_{-1} = 0 \) and \( j \in \{0, \cdots, K - 1\} \). Also, the long-term power constraint at the transmitter is expressed as \( \mathbb{E}_H \{P_{J(H)}\} \leq \text{SNR} \). Since in (4.4) we allocate \( P_j \) for all \( P_R(H) \in (P_{j-1}, P_j] \), we can view this mapping as a scalar quantizer of \( P_R(H) \), where each reconstruction point coincides with the upper boundary.

![Figure 4.2: A heuristic quantizer structure.](image)

In order to utilize a noisy feedback channel, we first propose a heuristic scalar quantizer structure for \( P_R(H) \), and later on, we prove its optimality conditions. In Fig. 4.2, the quantizer boundaries are shown by \( \{Q_j\} \) and the reconstruction points
are represented by \( \{P_j\} \). The set of reconstruction points \( \{P_j\} \) includes the members of a power control codebook that is known to the transmitter/receiver sides. Among \( j = \{0, \cdots, K-1\} \), the receiver chooses index \( j \) if \( P_R(H) \in (Q_{j-1}, Q_j] \) and sends it to the transmitter through the feedback link. The appropriate index for the region \( P_R(H) > Q_{K-1} \) will be discussed in the following parts of this chapter. Note that we won’t claim the optimality of this structure in this section.

Each feedback index \( j \) at the receiver is first permuted to an index \( j' \) using a one-to-one mapping \( j' = \delta(j) \), and then each bit of \( j' \) is sequentially transmitted over the feedback link. Therefore, the feedback bits are binary representation of index \( j' \). We call the mapping block \( \delta(j) \) in conjunction with the required binary bit transformation as a bit-mapping scheme. At the transmitter side, a demapping block is used after decoding the feedback bits and converting them into feedback indices.

Note that the indices of the proposed power control codebook are not necessarily sequential in their binary representations. When the feedback is error-prone, the structures of the mapping/demapping blocks should affect the outage probability of the system. As we will discuss in the following sections, the structure of the optimal quantizer depends on the bit-mapping scheme that we use at the feedback link.

The output of the feedback link is an index \( i \) and the transmitter chooses its power control factor \( P_i \) upon receiving this index. The indices \( i \) and \( j \) may be different due to the noise in the feedback link. Assume that the feedback channel is a DMC defined by \( \lceil \log_2 K \rceil \) uses of a binary symmetric channel (BSC) with crossover probability \( 0 \leq \rho \leq 0.5 \), that is known a priori. Note that if \( \rho > 0.5 \), then the transmitter can simply flip the feedback bits and obtain better results. Therefore,
the case of $\rho > 0.5$ is not considered here. Moreover, $\rho = 0.5$ is equivalent to a no-CSIT condition. The conditional index transition probability matrix of the feedback link can be expressed by the elements of its $j'$-th row and $i'$-column as

$$p(i' | j') = \rho^{d(i', j')} (1 - \rho)^{\lceil \log_2 K \rceil - d(i', j')}$$

where $d(i', j')$ denotes the Hamming distance between the binary representations of $i'$ and $j'$.

4.2 Quantizer Optimization with the Heuristic Structure

Our goal in this section is to optimize $(\{Q_i\}, \{P_i\})$ for the quantizer structure in Fig. 4.2 to minimize the outage probability of the system under a long-term power constraint. We won’t claim the optimality of this structure in this section and the optimality discussions are postponed to the next section.

First, we start with the feedback index assignment for region $P_R(H) > Q_{K-1}$. In this region, regardless of the index $j$, transmission incurs outage. Therefore, the receiver must choose a feedback index that results in the least transmission power consumption, noting that outage is inevitable in this region. Note that when the feedback is noisy, the probability of receiving a correct index $j$ at the transmitter is greater than the probability of receiving any other index $i \neq j$. Since,

$$(1 - \rho)^{\lceil \log_2 K \rceil} \geq (1 - \rho)^{\lceil \log_2 K \rceil - n} \rho^n$$

for any $n < \lceil \log_2 K \rceil$ and $\rho \leq 0.5$. The left hand side of the latter inequality is the probability of receiving the correct index and the right hand side is the probability of having $n$-bit errors. Therefore, by assigning the index corresponding
to the least power level in the reconstruction codebook, i.e., index 0 to the region \( P_R(H) > Q_{K-1} \), we can minimize the most probable transmission power when feedback is noisy and outage is inevitable. \(^1\) Note that we cannot advise switching-off the transmitter, since this requires an additional index in the reconstruction codebook.

To proceed with the optimization process, let us define the complimentary outage probability at rate \( R \) and with the transmission power \( P \) as

\[
F(P) \doteq \Pr[I(P) \geq R]
\]

(4.7)

Using this definition, the outage probability of the proposed system can be expressed as

\[
P_{\text{out}} = \Pr[P_R(H) > Q_{K-1}] + \sum_{i=0}^{K-1} \Pr[i; P_R(H) \leq Q_{K-1}; \text{outage}]
\]

\[
\quad = [1 - F(Q_{K-1})] + \sum_{i=0}^{K-1} \left\{ p(i|0)[F(Q_i) - F(P_i)] + \sum_{j=i+1}^{K-1} p(i|j)[F(Q_j) - F(Q_{j-1})] \right\}
\]

(4.8)

Also, the average transmission power can be expressed as

\[
P_{\text{avg}} = \sum_{i=0}^{K-1} \left\{ \Pr[P_R(H) > Q_{K-1}] p(i|0) + \sum_{j=0}^{K-1} \Pr[P_R(H) \in (Q_{j-1}, Q_j)] p(i|j) \right\} P_i
\]

\[
\quad = \sum_{i=0}^{K-1} \left\{ [1 - F(Q_{K-1})] p(i|0) + \sum_{j=0}^{K-1} [F(Q_j) - F(Q_{j-1})] p(i|j) \right\} P_i
\]

(4.9)

The constrained outage minimization problem can then be stated as follows:

\[
\min_{\{P_i\}, \{Q_i\}} P_{\text{out}}
\]

\[
s.t., \quad P_{\text{avg}} - \text{SNR} \leq 0
\]

and

\[
P_i - Q_i \leq 0, \quad Q_{i-1} - P_i \leq 0
\]

(4.10)

\(^1\)A more formal proof of the optimality of the quantizer structure will be provided in the next section.
Lemma 1. For a given set of boundaries, \( \{Q_j\} \) in the heuristic structure, the reconstruction points of the optimal quantizer coincide with the upper boundaries at high-SNR. Namely, as \( \text{SNR} \to \infty \), the optimal solutions to the above optimization problem should satisfy: \( P^*_j = Q_j, \forall j \).

Proof. The proof is provided in Appendix A. \qed

As discussed in the proof of this lemma, the KKT analysis justifies the structure of our proposed quantizer. Generally speaking, however, we cannot claim the global optimality of the proposed quantizer unless we show the convexity of the optimization problem. The optimality of the quantizer structure will be shown in the next section using a different approach.

Using the proposed quantizer structure, we proceed with the design of the quantizer. Using Lemma 1, the outage probability of the proposed transmission scheme can be expressed as

\[
P_{\text{out}} = [1 - F(P_{K-1})] + \sum_{i=0}^{K-1} \sum_{j=i+1}^{K-1} p(i|j) [F(P_j) - F(P_{j-1})]
\]

(4.11)

and the average transmission power \( P_{\text{avg}} \) can be written as

\[
P_{\text{avg}} = \sum_{i=0}^{K-1} \left\{ [1 - F(P_{K-1})]p(i|0) + \sum_{j=0}^{K-1} [F(P_j) - F(P_{j-1})]p(i|j) \right\} P_i
\]

(4.12)

Now, the simplified power control codebook design objective can be expressed as

\[
\min_{\{P_i\}} P_{\text{out}}
\]

(4.13)

s.t. \( P_{\text{avg}} \leq \text{SNR} \)
In Section 4.5, we numerically solve the above constrained optimization problem. In the next section, we investigate the optimality conditions of the proposed quantizer structure and show that this structure is optimal given a certain class of bit-mapping schemes is utilized at the feedback link.

4.3 Optimal Quantizer Structure

The proposed quantizer structure of the previous section has the property that all the Voronoi regions, except for one, are contiguous intervals. For this particular quantizer structure, we optimized the reconstruction points and proved that at high SNR, the reconstruction points also define the boundaries. In this section, we find the optimal partitioning of the space of channel realizations that minimizes the outage probability under an average power constraint. In other words, we remove the contiguity assumption of the Voronoi regions and consider a general form of Voronoi regions as arbitrary unions of sets. The goal is to show that with erroneous feedback, under certain conditions, the proposed quantizer structure in (4.4) is, in fact, optimal.

4.3.1 Bit-Mapping for the Optimal Quantizer Structure

Generally speaking, the optimal quantizer structure depends on the mapping of quantization indices to binary bits (bit-mapping). In the subsequent sections, we will prove the optimality of the quantizer structure for a certain class of bit-mappings. More specifically, we assume that the bit-mapping in use satisfies the following properties:

$$\forall \ell < j : \sum_{k=j}^{K-1} p(k|j) > \sum_{k=j}^{K-1} p(k|\ell)$$  (4.14)
∀ℓ > j : \( \sum_{k=j}^{K-1} p(k|j) \geq \sum_{k=j}^{K-1} p(k|\ell) \) (4.15)

∀ℓ < j, ∀m \leq K − 1 : \( \sum_{k=m}^{K-1} p(k|j) \geq \sum_{k=m}^{K-1} p(k|\ell) \) (4.16)

Table 4.1: Quasi-grey bit-mapping for different quantizer dimensions.

<table>
<thead>
<tr>
<th>dimension</th>
<th>index vector</th>
<th>bits</th>
</tr>
</thead>
<tbody>
<tr>
<td>K = 2</td>
<td>[0 1]</td>
<td>b = 1</td>
</tr>
<tr>
<td>K = 3</td>
<td>[0 2 1]</td>
<td>b = 2</td>
</tr>
<tr>
<td>K = 4</td>
<td>[0 3 2 1]</td>
<td>b = 2</td>
</tr>
<tr>
<td>K = 6</td>
<td>[0 3 5 1 2 4]</td>
<td>b = 3</td>
</tr>
<tr>
<td>K = 8</td>
<td>[0 3 6 5 2 7 1 4]</td>
<td>b = 3</td>
</tr>
<tr>
<td>K = 10</td>
<td>[0 6 9 7 3 5 1 3 8 2]</td>
<td>b = 4</td>
</tr>
<tr>
<td>K = 12</td>
<td>[0 7 2 8 4 9 5 11 1 3 6 10]</td>
<td>b = 4</td>
</tr>
</tbody>
</table>

We call such a mapping scheme a quasi-grey bit-mapping and we conjecture that a quasi-grey bit-mapping exists for an arbitrary number of quantization regions, \( K \). In the case of \( K = 4 \), for instance, a quasi-grey bit-mapping can be realized as

\[
\begin{align*}
\text{reconstruction points} & \rightarrow & \mathcal{P}_0 & \quad \mathcal{P}_1 & \quad \mathcal{P}_2 & \quad \mathcal{P}_3 \\
\downarrow & & \downarrow & & \downarrow & (4.17) \\
\text{quasi-grey bit-mapping} & \rightarrow & 00 & 11 & 10 & 01
\end{align*}
\]

The index transition probability matrix of the associated DMC can be written as

\[
p_{ij} = \begin{pmatrix}
(1 - \rho)^2 & \rho^2 & \rho(1 - \rho) & \rho(1 - \rho) \\
\rho^2 & (1 - \rho)^2 & \rho(1 - \rho) & \rho(1 - \rho) \\
\rho(1 - \rho) & \rho(1 - \rho) & (1 - \rho)^2 & \rho^2 \\
\rho(1 - \rho) & \rho(1 - \rho) & \rho^2 & (1 - \rho)^2
\end{pmatrix} (4.18)
\]

It is easy to show that properties (6.3)-(6.4) hold for the transition probability matrix (4.18).
A quasi-grey bit-mapping for an arbitrary $K$ may not be unique and it can be found by an exhaustive search. Table 4.1 shows the search outcome for different values of $K$. In this table, such a mapping can be obtained from the $b$-bit binary representation of the elements of the index vector introduced in the table, where $b = \lceil \log_2 K \rceil$.

### 4.3.2 Optimal Quantizer Structure

Recall that for a specific realization of the fading channel $H$, the minimum outage-free power is denoted by $P_R(H)$ and is given as the solution of (4.3). Suppose that the optimal size $K$ power codebook is given as $\{P_j\} = \{P_0, P_1, \cdots, P_{K-1}\}$, where we assume, without loss of generality

$$0 \leq P_0 \leq P_1 \leq \cdots \leq P_{K-1} < \infty \quad (4.19)$$

Now denote the optimal partitioning of the channel space by

$$\Psi^o = \{ \psi^o_0, \psi^o_1, \cdots, \psi^o_{K-1} \} \quad (4.20)$$

In other words, the optimal quantization scheme at the receiver assigns feedback index $j$ to the channel realizations in partition $\psi^o_j$. The following theorem shows that under certain conditions, the proposed quantizer structure of Section 4.1.2 is optimal almost everywhere.

For simplicity, we use the term “zero-probability set” to indicate channel realizations that occur with zero probability.
Theorem 3. Suppose that a quasi-grey bit-mapping is used for the $K$-level quantizer. In the high SNR regime, $\psi^*_j - \psi^*_0$ has zero probability, where

$$
\psi^*_j = \{H; P_{j-1} < P_R(H) \leq P_j\}, \ j \in \{1, \cdots, K - 1\} \tag{4.21}
$$

$$
\psi^*_0 = \{H; 0 \leq P_R(H) \leq P_0\} \cup \{H; P_{K-1} < P_R(H)\} \tag{4.22}
$$

Proof. The proof is provided in Appendix C.  \footnotemark

\footnotetext{The pre-requisites of this proof are provided in Appendix B-1.}

4.4 Asymptotic Analysis

In this section, the goal is to characterize the behavior of the outage probability of the proposed system at very high-SNR. In [53]-[56], Kim et al. determined the high-SNR behavior of the system with error-free feedback using the definition of the diversity gain,

$$
d = \lim_{\text{SNR} \to \infty} \frac{-\log \text{P}_{\text{out}}}{\log \text{SNR}} \tag{4.23}
$$

They proved that with error-free feedback and for a fixed-rate system, the diversity gain can be expressed as $d = \sum_{k=0}^{K-1}(d_0)^{k+1}$, where $d_0 = rt$ [53–56]. The polynomial diversity gain increase with the number of quantization bins of the feedback information was also shown in [59]. Here, we study the diversity gain of the proposed closed-loop transmission scheme with noisy feedback.

Theorem 4. The diversity gain of the proposed transmission scheme, with power control based on noisy quantized feedback is equal to the diversity gain of a no-CSIT
system. That is, when $\rho > 0$, we have

$$d = rt$$

(4.24)

where $r$ and $t$ are the number of receive antennas and the number of transmit antennas, respectively.

Proof. The proof is provided in Appendix D.\footnote{The pre-requisites of this proof are provided in Appendix B-2.}

4.5 Numerical Results

In this section, we numerically solve the outage minimization problem (4.13) based on the optimal quantizer structure. For future references, note that over multiple-input single-output (MISO) and single-input multiple-output (SIMO) (thus SISO) Rayleigh fading channels, the complimentary outage probability $F(P_j)$ can be derived in closed-form. For example, it is straightforward to show that over MISO channels with $t$ transmit antennas, $F(P_j) = 1 - \frac{\Gamma\left(\Gamma\left(\frac{\exp(R) - 1}{\rho_j}\right)\right)}{(r-1)!}$, where $\Gamma(a, x) = \int_0^x u^{a-1}e^{-u}du$ is the incomplete gamma function. Similarly, over SIMO channels with $r$ receive antennas, $F(P_j) = 1 - \frac{\Gamma\left(\Gamma\left(\frac{\exp(R) - 1}{\rho_j}\right)\right)}{(r-1)!}$. In the MIMO channel case, however, $F(P_j)$ cannot be obtained in closed-form and one way of calculating this probability is through Monte Carlo method.

In the first example, we solve (4.13) for a SISO link with $K = 2$ and $K = 4$ feedback regions. Fig. 4.3 shows the outage probability versus the average SNR of this system. We can see that the no-CSIT curve shows diversity gain $d_0 = 1$ that is attributed to the diversity gain of a fixed-rate and no-CSIT system. Moreover, the noiseless feedback system shows diversity gain $d = 2$ for $K = 2$ and $d = 4$ for $K = 4$.\footnote{The pre-requisites of this proof are provided in Appendix B-2.}
which coincides with the polynomial diversity gain $d = \sum_{k=0}^{K-1} d^k_{k+1}$ of a fixed-rate system [53]-[56]. Also note that as we increase the number of quantization regions, we achieve lower outage probabilities.

Another observation that we can make from this figure is that the diversity gain of the noisy feedback system is the same as the diversity gain of a no-CSIT scheme. However, the outage probability of the noisy feedback system is less than that of the no-CSIT scheme and this superiority is pronounced more for lower feedback error probabilities. Therefore, although noisy feedback couldn’t offer any additional diversity gain compared to a no-CSIT scheme, it improved the system.
Fig. 4.4 shows the outage probability of a 2x2 MIMO channel with $K = 2$ feedback regions. Again, the diversity gain of the noiseless feedback system polynomially increases with $K$ and noisy feedback and no-CSIT systems show the same diversity gain. Let us emphasize that noisy feedback doesn’t increase the diversity gain of the system. It nevertheless improves the system performance compared to a no-CSIT scenario. Therefore, feedback information is useful even with errors at the feedback link. From this figure, we can also see that the outage performance of the system converges to that of a no-CSIT scheme, when the feedback link degrades.
severely or when the cross-over probability of the BSC approaches 0.5. However, the system is never inferior to a no-CSIT scheme. We attribute this property of the system to the structure of the optimal quantizer that we will investigate in more detail in the next example.

In this example, we study a $2 \times 1$ MISO system with $K = 4$ feedback regions. The optimal quantizer structure can be obtained through solving (4.13). Fig. 4.5 shows the structure of the optimal quantizer for the above system. The leftmost set of points denote the optimal quantizer structure with error-free feedback and the rightmost one is the no-CSIT solution. As the error probability of the feedback
Figure 4.6: $P_{\text{out}}$ vs SNR for a $2 \times 1$ MISO channel, with $K = 4$ regions, and the transmission rate $R = 6$ nats/channel use. Identity index mapping is compared to the optimal index assignment.

link increases, the reconstruction points of the optimal power control codebook merge and the optimal quantizer solution converges to the no-CSIT solution. This property of the optimal quantizer resembles channel optimized scalar quantizers (COSQs) [15, 24, 61]. In other words, for more noisy channels, the Voronoi regions of the optimal quantizer will shrink and in some cases, the number of resolvable Voronoi regions will diminish.

Note that COSQ is a joint source-channel coding module. In a COSQ structure, source coding and channel coding should interact in their impact on the feedback information. As the cross-over probability of the feedback link increases,
the reconstruction points of the power codebook merge and some of the Voronoi regions shrink or vanish. Therefore, with feedback errors, the system degenerates to a closed-loop system with fewer feedback regions (less source-coding rate), although the number of feedback bits are fixed, thus the channel coding role of the COSQ becomes more prominent. This structure is more general than separate quantization and coding of the feedback bits across the reverse link, and thus it should perform better than separation, when the same feedback channel resources are used.

Next, to show the role of bit-mapping in the optimal quantizer structure, we optimized the system using a (sub-optimal) identity bit-mapping, and we also considered the (optimal) quasi-grey bit-mapping scheme. In both cases, we used the proposed circular quantizer structure with contiguous Voronoi regions. According to Fig. 4.5, the outcomes of the optimization, i.e., the optimal reconstruction points are different for these two cases. Therefore, we conclude the bit-mapping scheme that we choose at the feedback channel does change the structure of the optimal quantizer. The circular and contiguous quantizer structure is only optimal for a quasi-grey bit-mapping scheme, and for other mappings, some other structure may be optimal.

Fig. 4.6 shows the outage probability of the above 2x1 MISO system with $K = 4$ feedback regions. The same diversity results and outage behavior as the results of the previous examples can be also observed here. Moreover, the amount of performance improvement that (optimal) quasi-grey bit-mapping with circular and contiguous quantizer structure can provide compared to the (sub-optimal) identity mapping can be observed in this figure. Note also that for higher quality feedback links, the gain of optimal bit-mapping is more noticeable.
In this figure, we also compare the performance of the power control scheme that is the focus of this part of the thesis to the popular Grassmannian beamforming that is a limited-rate feedback scheme using channel eigenvector information at the transmitter [70, 75]. We see that noiseless beamforming outperforms a no-CSIT system with the same diversity gain. However, even with a small feedback error probability, the performance of Grassmannian beamforming degrades severely, and becomes even worse than that of a no-CSIT system. This behavior is not surprising, since Grassmannian beamforming is designed for an error-free feedback system. In the light of these observations, we claim that power control is more effective and more robust compared to Grassmannian beamforming. Since as shown in Fig. 4.6, even noiseless beamforming cannot outperform the optimal power control scheme with a moderate feedback error probability. Of course, the benefits of power control come at a price, which is the complexity of power adaptation at the transmitter, where beamforming can be performed only with a fixed power. However, if there is any possibility of errors in the feedback link, power control becomes more attractive than beamforming to minimize the outage probability of the system.

4.6 Conclusions

In this chapter, we formulated and optimized the outage probability of transmission over quasi-static fading channels with temporal power control, when a noisy version of quantized channel state information is available at the transmitter. The presented scheme bypassed the requirement of receiver knowledge of the noisy feedback index at the transmitter.

At high-SNR, using a quasi-grey bit-mapping scheme at the feedback link, the
optimal power control codebook with a noisy feedback channel belongs to a circular quantizer structure with contiguous Voronoi regions. Also, the optimal quantizer follows a structure similar to channel optimized scalar quantizers. The outage probability of this system is less than that of a no-CSIT scheme and therefore noisy feedback is useful. However, asymptotic analysis of the outage probability shows that noisy feedback cannot offer any additional diversity gain compared to a no-CSIT scheme, if the error probability of feedback doesn’t vanish with the SNR.
4.7 Appendix

4.7.1 Proof of Lemma 1

Proof. Our goal is to investigate the Karush-Kuhn-Tucker (KKT) optimality conditions. Problem (4.10) can be simplified using the Lagrange multipliers $\lambda_p$, $\lambda^l_i$, $\lambda^u_i$, and the Lagrangian

$$
\mathcal{L} = P_{\text{out}} + \lambda_p (P_{\text{avg}} - \text{SNR}) + \sum_{i=0}^{K-1} \left[ \lambda^l_i (Q_{i-1} - P_i) + \lambda^u_i (P_i - Q_i) \right]
$$

(4.25)

Here, the KKT optimality conditions can be written as [7]

$$
-p(i|i) f(P_i) + \lambda_p p(i) - \lambda^l_i + \lambda^u_i = 0 \quad (i)
$$

$$
\lambda^l_i \geq 0 \quad \text{and} \quad \lambda^u_i \geq 0 \quad (ii)
$$

$$
\lambda^u_i (P_i - Q_i) = 0 \quad \text{and} \quad \lambda^l_i (Q_{i-1} - P_i) = 0 \quad (iii)
$$

(4.26)

for $\forall i$, where $f(P_i)$ is the derivative of $F(P)$ at $P_i$ and $p(i)$ is the marginal probability of index $i$ at the transmitter. Note that $F(P)$ is a non-decreasing function of $P$. Therefore, $f(P_i)$ is always non-negative. Moreover, as $\text{SNR} \to \infty$, the optimization problem (4.10) approaches an un-constrained-power optimization problem and thus the Lagrange multiplier associated with the power constraint approaches zero, i.e., $\lim_{\text{SNR} \to \infty} \lambda_p = 0$. Since the first and the second terms in (i) are negative and zero in the limit of high-SNR, respectively and $\lambda^l_i \geq 0$, we must have $\lambda^u_i > 0$. Therefore from relation (iii), we conclude that $P_i - Q_i = 0$. In other words, at high-SNR, the structural constraint $P_i - Q_i \leq 0$ is active. $\square$

4.7.2 Pre-requisites for the proofs of the Theorems

In this section, we will present a few results that will be used in the proofs of the theorems.
pre-requisites of Theorem 1

Lemma 2. Suppose that the derivative of the complimentary outage probability function $F(x)$ in (4.7) is represented by $f(x)$. In the optimal power codebook, we have

$$
\lim_{\text{SNR} \to \infty} f(\mathcal{P}_{K-1}) \mathcal{P}_{K-1} = 0
$$

Proof. First we prove a general result for a generic function and then we apply the result to prove the lemma. Suppose that a function $h(x)$ is defined for $\forall x > 0$ and it is bounded above by zero, i.e., $h(x) \leq 0$, $\forall x > 0$. Moreover, suppose that the derivative of $h(x)$, that is $h'(x)$ exists over $\forall x > 0$. We want to show that if the limit of $h'(x)$ also exists, it cannot be a positive number. In other words, $\lim_{x \to \infty} h'(x) \leq 0$.

We use contradiction. Let us suppose the contrary, that is

$$
\lim_{x \to \infty} h'(x) = m > 0 \Leftrightarrow \forall \delta > 0, \exists N > 0; x > N \Rightarrow |h'(x) - m| < \delta \quad (4.27)
$$

Pick $x_0 > N$ and $m_0 < m$ and find $d_0$, such that

$$
h(x_0) = m_0 x_0 + d_0 \quad (4.28)
$$

Now, we must prove that

$$
\forall x > x_0, \ h(x) > m_0 x + d_0 \quad (4.29)
$$

Again, suppose the contrary, that is

$$
\exists x_1 > x_0; \ h(x_1) \leq m_0 x_1 + d_0 \quad (4.30)
$$

Then, according to (4.28), (4.30), and the mean value theorem, we have:

$$
\exists x' \in (x_0, x_1); \ h'(x') \leq m_0 \quad (4.31)
$$
This is clearly in contradiction to (4.27). This leads us to show that (4.29) is true. If (4.29) is true, then the function \( h(x) \) is greater than a linear function for \( x > N \) and cannot be bounded. This contradicts the assumption that we began with. To summarize, we showed that if \( h(x) \) is bounded above and the limit of its derivative exists, \( \lim_{x \to \infty} h'(x) \leq 0 \).

Now, define \( h(x) \triangleq F(x)x - x \), where \( F(\cdot) \) is the complementary outage probability function, defined in (4.7). Note that \( F(x) \leq 1 \), therefore, \( h(x) \leq 0 \). Now, suppose that the derivative of \( h(x) \) exists \(^4\). From the results of the previous argument, we know the following about the derivative of \( h(x) \):

\[
\lim_{x \to \infty} f(x)x + F(x) - 1 \leq 0 \tag{4.32}
\]

Moreover, we know that \( f(x)x \) is non-negative and \( \lim_{x \to \infty} F(x) = 1 \). The latter two facts and (4.32) result in

\[
\lim_{x \to \infty} f(x)x = 0 \tag{4.33}
\]

To proceed, we use the result of Lemma 3, in the next sub-section, which states that \( P_{K-1} \to \infty \) as \( \text{SNR} \to \infty \). Since \( F(P_{K-1}) \leq 1 \), we have \( F(P_{K-1})P_{K-1} \leq P_{K-1} \). Therefore, we can define the bounded function

\[
h(P_{K-1}) = F(P_{K-1})P_{K-1} - P_{K-1} \tag{4.34}
\]

Moreover, \( f(P_{K-1})P_{K-1} \) is always non-negative and \( \lim_{P_{K-1} \to \infty} F(P_{K-1}) = 1 \). We can then apply (4.33) to show that

\[
\lim_{P_{K-1} \to \infty} f(P_{K-1})P_{K-1} = 0 \tag{4.35}
\]

Therefore, the result of the lemma follows. \( \square \)

\(^4\)Note that for multiple-antenna Rayleigh fading channels, \( \lim_{p \to \infty} \frac{d}{dp} (F(p)p - p) = \lim_{p \to \infty} f(p)p + F(p) - 1 \) exists.
Pre-requisites of Theorem 2

Without loss of generality, throughout the rest of the proofs, we assume that

\[ 0 \leq \mathcal{P}_0 \leq \mathcal{P}_1 \leq \cdots \leq \mathcal{P}_{K-1} < \infty \]  \quad (4.36)

**Lemma 3.** *In the optimal power codebook, \( \mathcal{P}_0 \leq \text{SNR} \) and \( \mathcal{P}_{K-1} \geq \text{SNR} \).*

**Proof.** We prove this lemma by contradiction. First, assume that \( \mathcal{P}_0 > \text{SNR} \). Since \( \mathcal{P}_i \geq \mathcal{P}_0, \forall i = \{1, \cdots, K - 1\} \), we conclude that \( \mathcal{P}_i > \text{SNR}, \forall i \), and the power constraint (4.12) will be violated. Second, assume that \( \mathcal{P}_{K-1} < \text{SNR} \). Since \( \mathcal{P}_i \leq \mathcal{P}_{K-1}, \forall i = \{0, \cdots, K - 2\} \), we conclude that \( \mathcal{P}_i < \text{SNR} \forall i \). The latter inequality requires the expected power of the system to be less than the SNR. Since the outage probability is a decreasing function of the expected power, with this assumption, the outage probability of the system would be greater than that of a system with the expected power SNR and this violates the optimality of the power control codebook. \( \square \)

The following corollary is a direct result of Lemma 3. **Corollary 1-** In the optimal power control codebook, there should be an intermediate index \( \varsigma \in \{0, \cdots, K - 2\} \), such that

\[ \mathcal{P}_0 \leq \cdots \leq \mathcal{P}_\varsigma \leq \text{SNR} \leq \mathcal{P}_{\varsigma+1} \leq \cdots \leq \mathcal{P}_{K-1} \]  \quad (4.37)

### 4.7.3 Proof of Theorem 3

**Proof.** Let \( \psi_{0l}^* = \{H; 0 \leq \mathcal{P}_R(H) \leq \mathcal{P}_0\} \) and \( \Psi_u = \{H; \mathcal{P}_R(H) > \mathcal{P}_{K-1}\} \). Note that the random variable being quantized is the scalar quantity \( \mathcal{P}_R(H) \), and as a result, \( \Psi_u, \psi_{0l}^*, \) and \( \psi_j^* (1 \leq j \leq K - 1) \) each represent a single contiguous interval on the real line. Furthermore, note that once we prove that \( \psi_j^* - \psi_j^0 \) has
zero probability, this shows the optimality of the proposed quantizer, since both 
\( \{ \psi^*_k \}_{k=0}^{K-1} \) and \( \{ \psi^o_k \}_{k=0}^{K-1} \) span the entire space of channel realizations.

We first claim that all channel realizations in \( \Psi_u \) belong to a single Voronoi region and that the optimal index for this region is the index \( j = 0 \) that minimizes the power consumption. To prove this, we note that the channel realizations in this region will certainly experience outage, regardless of the index \( i \) appearing at the transmitter. Suppose that we allocate index \( \ell > 0 \) to this region. The average power consumption with this allocation, given \( \Psi_u \) occurs, can be written as

\[
P_{\text{avg}}^\ell = \sum_{k=0}^{K-1} p_k |\ell | P_k,
\]

whereas upon assigning index 0 to \( \Psi_u \), the corresponding parameter can be written as

\[
P_{\text{avg}}^0 = \sum_{k=0}^{K-1} p_k |0 | P_k.
\]

According to the third property of the quasi-grey bit mappings, (6.4), for \( m = 0 \), and noting that \( \{ P_j \} \) is a non-decreasing sequence, we can see that

\[
P_{\text{avg}}^0 \leq P_{\text{avg}}^\ell.
\]

Therefore, it is more power efficient to assign index 0 to \( \Psi_u \). We therefore conclude that in the optimal structure, the index assigned to this partition is \( j = 0 \).

For simplicity, in the rest of the proof, we exclude \( \Psi_u \) from the possible channel realizations, and we only discuss the complement of \( \Psi_u \). Moreover, unless otherwise stated, we only consider the subset \( \psi^o_0 \) of \( \psi^*_0 \) when dealing with index \( j = 0 \).

Recall that the optimal set of channel realizations mapped to a given index \( j \) is denoted by \( \psi^o_j \) for \( j \in \{0, 1, \cdots, K - 1\} \). In its most general form, \( \psi^o_j \) can be any arbitrary union of non-overlapping intervals, besides some sets of zero probabilities (discrete realizations). We only consider the subsets of \( \psi^o_j \) with non-zero probabilities. Our goal is to show that \( \psi^o_j \) is equivalent to \( \psi^*_j \) in a probabilistic sense and as such, it is comprised of a single interval on the real line.\(^5\)

We proceed the proof of the theorem by contradiction. Suppose that for some

\(^5\)With an exception of \( \psi^o_0 \) that is a union of two non-overlapping sets defined in (4.22).
\( j \), the set \( \psi_j - \psi_j^o \) has a non-zero probability. This set represents all the channel realizations mapped to index \( j \) in the proposed quantizer that don’t belong to \( \psi_j^o \). Such channel realizations, therefore, belong to some \( \psi_k^o, k \neq j \), since \( \{ \psi_k^o \}_{k=0}^{K-1} \) spans the entire space of channel realizations. This, in turn, requires at least one of the two sets \( \psi_j^- \) or \( \psi_j^+ \) to have a non-zero probability, where

\[
\psi_j^- = (\psi_j^* - \psi_j^o) \cap \left( \bigcup_{k=0}^{j-1} \psi_k^o \right) \quad (4.38)
\]

\[
\psi_j^+ = (\psi_j^* - \psi_j^o) \cap \left( \bigcup_{k=j+1}^{K-1} \psi_k^o \right) \quad (4.39)
\]

To show the contradiction, we need to prove that in the optimal quantizer structure, the above two sets must have zero probabilities.

In the first part of the proof, we use contradiction to show that \( \psi_j^- \) has zero probability. Assume the contrary. Then (4.38) requires \( \psi_j^- \) to have a non-zero probability subset of \( \bigcup_{k=0}^{j-1} \psi_k^o \) with elements in \( \psi_{\ell}^o \) for some \( \ell < j \). We can then arbitrarily choose elements of such a subset to create a subset denoted by \( \psi_{j\ell}^- \) with an arbitrarily small non-zero probability. In the following, we show that by changing the optimal index \( \ell \) of \( \psi_{j\ell}^- \) and making some adjustments in the power codebook, the overall outage probability of the system can be held constant while the power consumption can be reduced. This will then contradict the optimality of the quantizer. This is accomplished in two steps. In the first step, the outage probability is reduced by re-indexing \( \psi_{j\ell}^- \) and this results in an increase in the power consumption. In the second step, we revert the outage probability to its original value by making a modification in the power codebook and show that this modification reduces the power consumption by an amount more than the power increase of the first step.

**Step 1:** Suppose that we change the index of region \( \psi_{j\ell}^- \) from \( \ell \) to \( j \). This
means that all channel realizations in $\psi_{j\ell}^-$ will now be assigned index $j$ at the receiver. Since $\psi_{j\ell}^- \subset \psi_j^*$ and $\mathcal{P}_j$ is a non-decreasing sequence according to (4.36), then any index $k$ received at the transmitter for $\psi_{j\ell}^-$ will result in outage if $k < j$.

Thus, $\sum_{k=j}^{K-1} p(k|j)$ and $\sum_{k=j}^{K-1} p(k|\ell)$ denote the non-outage probabilities after and before re-indexing, respectively, when the channel realization falls in $\psi_{j\ell}^-$. The quasi-grey bit-mapping property (6.3) indicates that re-indexing reduces the outage probability of the system by

$$\Delta^1_{\text{out}} = \Pr(\psi_{j\ell}^-) \left( \sum_{k=j}^{K-1} p(k|j) - \sum_{k=j}^{K-1} p(k|\ell) \right) = \Pr(\psi_{j\ell}^-) C_1$$

for some $0 < C_1 < \infty$, where $\Pr(\psi_{j\ell}^-)$ is the probability of the channel realization being in $\psi_{j\ell}^-$. The above re-indexing changes the expected power consumption of the system for region $\psi_{j\ell}^-$ from $\Pr(\psi_{j\ell}^-) \sum_{k=0}^{K-1} p(k|\ell) \mathcal{P}_k$ to $\Pr(\psi_{j\ell}^-) \sum_{k=0}^{K-1} p(k|j) \mathcal{P}_k$. The latter difference is given by

$$\Delta^1_{\text{avg}} = \Pr(\psi_{j\ell}^-) \sum_{k=0}^{K-1} (p(k|j) - p(k|\ell)) \mathcal{P}_k$$

that is a positive number according to (6.4) and the non-decreasing property of the power codebook, (4.36). By using the absolute value of the terms $p(k|j) - p(k|\ell)$ and the largest power level, $\mathcal{P}_{K-1}$, in the above equation, we can upper bound $\Delta^1_{\text{avg}}$ as

$$\Delta^1_{\text{avg}} \leq \Pr(\psi_{j\ell}^-) C_2 \mathcal{P}_{K-1}$$

for some $0 < C_2 < \infty$. From (4.40), we can then conclude that

$$\Delta^1_{\text{avg}} \leq C_3 \mathcal{P}_{K-1} \Delta^1_{\text{out}}$$

for some $0 < C_3 < \infty$. Thus, the net effect of re-indexing was to reduce the outage probability and to increase the expected power by the amounts that are related through (4.43).
Step 2: In this step, we show that by changing the quantizer structure, it is possible to 1) revert the outage probability to its original value before the re-indexing of step 1 and 2) save the expected power by an amount more than the increase in the power consumption of step 1, (4.43). This will then contradict the optimality of the quantizer. For this purpose, we slightly reduce $P_{K-1}$ to $P'_{K-1} = P_{K-1} - \epsilon$, for some $\epsilon > 0$, and investigate the resulting effects on the outage versus expected power performance of the system.

We assume that $\epsilon$ is sufficiently small so that all the channel realizations in $\mathcal{P}(H) \in (P'_{K-1}, P_{K-1}]$ are assigned to a single index $m \in \{0, 1, \cdots, K-1\}$. We denote the marked region by $\psi_{K-1,m}$. This power reduction increases the outage probability of the system by

$$\Delta^2_{\text{out}} = \Pr(\psi_{K-1,m}) p(K-1|m) = \left[ F(P_{K-1}) - F(P'_{K-1}) \right] p(K-1|m) \quad (4.44)$$

while reducing the power consumption by

$$\Delta^2_{\text{avg}} = \Pr[K-1] (P_{K-1} - P'_{K-1}) \quad (4.45)$$

where $\Pr[K-1]$ is the probability that index $K-1$ appears at the transmitter. It is important to note that in a noisy feedback system, $\Pr[K-1] \neq 0$.

By equating the changes in the outage probability of the two steps, i.e., $\Delta^1_{\text{out}} = \Delta^2_{\text{out}}$, and from (4.43), we arrive at the following upper bound for the power increase of step 1:

$$\Delta^1_{\text{avg}} \leq C_3 \left[ F(P_{K-1}) - F(P'_{K-1}) \right] p(K-1|m) P_{K-1} \quad (4.46)$$

We now show that in the high SNR regime, $\Delta^2_{\text{avg}}$ is greater than the above upper bound for $\Delta^1_{\text{avg}}$. To prove this, (4.45) and (4.46) require that

$$\Pr[K-1] (P_{K-1} - P'_{K-1}) > C_3 \left[ F(P_{K-1}) - F(P'_{K-1}) \right] p(K-1|m) P_{K-1} \quad (4.47)$$
or equivalently,

\[
\frac{1}{c_3} \frac{\Pr[K-1]}{p(K-1|m)} > \frac{[F(P_{K-1}) - F'(P_{K-1})]}{(P_{K-1} - P'_{K-1})} P_{K-1} \tag{4.48}
\]

The left hand side of (4.48) is a bounded, positive real number denoted by \( C_4 \equiv \frac{1}{c_3} \frac{\Pr[K-1]}{p(K-1|m)} \). Since both \( \Pr[K-1] \) and \( p(K-1|m) \) are non-zero real numbers. For an arbitrarily small \( \epsilon \), the right hand side of (4.48) converges to \( f(P_{K-1}) P_{K-1} \), where \( f(x) \) is the derivative of \( F(x) \), assuming that the derivative exists. To prove the inequality, thus, we need to show that

\[
C_4 > f(P_{K-1}) P_{K-1} \tag{4.49}
\]

for any given \( C_4 \). Corollary 1 shows that \( P_{K-1} \to \infty \) in the high SNR regime, and Lemma 2 consequently proves (4.49), since the limit of the right hand side of (4.49) is zero. We can now conclude from (4.45) and (4.46) that

\[
\Delta_2^{P_{avg}} > \Delta_1^{P_{avg}} \tag{4.50}
\]

In other words, the two steps have kept the outage probability intact, while reducing the expected power of the system. This contradicts the optimality of the partitioning \( \{\psi^0_j\} \) and the power codebook \( \{P^*_j\} \).

In the second part of the proof, we show that in the optimal quantizer structure, \( \psi^+_j \), defined in (4.39) also has zero probability. We use contradiction once again. Suppose that for some \( \ell > j \), the set \( \psi^+_j \cap \psi^0_{\ell} \) has a non-zero probability. We change the index \( \ell \) of a subset of this set, denoted by \( \psi^+_j \), to \( j \). According to (4.15), \( \sum_{k=0}^{K-1} p(k|j) \geq \sum_{k=0}^{K-1} p(k|\ell) \), and as a result, re-indexing of \( \psi^+_j \) reduces the outage probability of the system. Furthermore, according to (6.4) and the non-decreasing property of the power control codebook, we know that

\[
\sum_{k=0}^{K-1} p(k|\ell) P_k \geq \sum_{k=0}^{K-1} p(k|j) P_k \tag{4.51}
\]
Therefore, re-indexing also reduces the average power consumption. Therefore, index $\ell$ cannot be the optimal index assigned to the region $\psi_{j\ell}^+$ and the contradiction follows. In conclusion, in the optimal quantizer structure, we cannot have a non-zero probability set $\psi_{j\ell}^+ \subset \psi_j^+ \cap \psi_\ell^o$ for any $\ell > j$.

To conclude the proof, note that the above two steps show that $\Psi_j^o \subset \Psi_j^*$ holds with possible exceptions on some zero-probability subsets of $\Psi_j^*$. Moreover, since both $\{\Psi_j^*\}$ and $\{\Psi_j^o\}$ span the entire space of channel realizations, we conclude that $\Psi_j^* \subset \Psi_j^o$. Therefore, $\Psi_j^o = \Psi_j^*$ holds with possible exceptions on some subsets with zero probabilities.

4.7.4 Proof of Theorem 4

Let us adopt the notation

$$f(\text{SNR}) = \text{SNR}^b \Leftrightarrow \lim_{\text{SNR} \to \infty} \frac{\log f(\text{SNR})}{\log \text{SNR}} = b$$ (4.52)

as the definition of the order of a function of SNR. Operators $\leq$ and $\geq$ can also be defined similarly.

Proof. First, we show that the diversity gain is upper bounded as $d \leq rt$, using the result of the Corollary 1. Suppose that the intermediate index introduced in Corollary 1 is $\varsigma = 0$. In this case, $\lim_{\text{SNR} \to \infty} F(\mathcal{P}_\kappa) = 1$, $\kappa \in \{1, \cdots, K-1\}$ and from (4.11), the limit of the system outage probability can be expressed as

$$\lim_{\text{SNR} \to \infty} P_{\text{out}} = \lim_{\text{SNR} \to \infty} p(0|1)[1-F(\mathcal{P}_0)]$$ (4.53)

Now, suppose that $\varsigma = 1$. Then, $\lim_{\text{SNR} \to \infty} F(\mathcal{P}_\kappa) = 1$, $\kappa \in \{2, \cdots, K-1\}$ and the limit of the system outage probability becomes

$$\lim_{\text{SNR} \to \infty} P_{\text{out}} = \lim_{\text{SNR} \to \infty} p(0|1)[F(\mathcal{P}_1)-F(\mathcal{P}_0)] + p(0|2)[1-F(\mathcal{P}_1)] + p(1|2)[1-F(\mathcal{P}_1)]$$ (4.54)
\[
\geq \lim_{\text{SNR} \to \infty} p'_\text{min} \left( F(P_1) - F(P_0) + 1 - F(P_1) + 1 - F(P_1) \right)
\]
\[
= \lim_{\text{SNR} \to \infty} p'_\text{min} \left( [1 - F(P_0)] + [1 - F(P_1)] \right)
\]
\[
\geq \lim_{\text{SNR} \to \infty} p'_\text{min}[1 - F(P_0)]
\]
(4.55)

where \( p'_\text{min} = \min\{p(0|1), p(0|2), p(1|2)\} \). Note that if \( \rho > 0 \), the term \( p'_\text{min} \) is a positive real number.

Repeating the above arguments for \( 2 < \varsigma \leq K - 2 \), we observe that the high-SNR expression of the outage probability is always greater than or equal to \( p'[1 - F(P_0)] \), where \( p' \) is a positive and constant number. Therefore, if \( \rho \) doesn’t vanish in the limit of high-SNR, we have

\[
P_{\text{out}} \geq 1 - F(P_0)
\]
(4.56)

Note that \( 1 - F(P_0) \) is equivalent to the outage probability of a no-CSIT system with average power \( P_0 \leq \text{SNR} \). Therefore, from [103], we have

\[
1 - F(P_0) \geq \text{SNR}^{-rt}
\]
(4.57)

From (4.57), we conclude that the diversity gain of the noisy feedback system is bounded by

\[
d \leq rt
\]
(4.58)

Now we prove that the diversity gain \( d = rt \) is achievable using an example. Suppose that in a hypothetical power control codebook, \( P_i = \text{SNR}, \forall i = \{0, \cdots, K - 1\} \). The latter codebook doesn’t violate the power constraint in (4.12) and is a valid (but not necessarily optimal) solution. Moreover, since with this codebook \( \lim_{\text{SNR} \to \infty}[F(P_i) - F(P_j)] = 0, \forall i, j \), from (4.11), the high-SNR outage probability of this scheme can be expressed as

\[
\lim_{\text{SNR} \to \infty} P_{\text{out}} = \lim_{\text{SNR} \to \infty} [1 - F(P_{K-1})]
\]
As a result, we can derive the diversity gain of the system with the latter codebook as

\[
\lim_{\text{SNR} \to \infty} \frac{-\log P_{\text{out}}}{\log \text{SNR}} = \lim_{\text{SNR} \to \infty} \frac{-\log[1 - F(\text{SNR})]}{\log \text{SNR}} = rt
\]

(4.59)

noting that the two limits in the above expression exist. Therefore, the diversity gain \( d = rt \) can be achieved using the above codebook. The outage probability of the system employing the optimal power codebook must be less than or equal to the achievable outage probability with the above codebook. Therefore, we have a lower bound on the diversity gain of the optimal system,

\[
d \geq rt
\]

(4.60)

From (4.58) and (4.60) we conclude that \( d = rt \).
Chapter 5

Joint Source-Channel Coding for Quasi-Static Fading Channels with Noisy Quantized Feedback

Recent demand for multimedia transmission over wireless channels has motivated the problem of transmitting an analog source over a fading channel. In this context, Shannon’s separation theorem states that the source and channel coding tasks can be done separately when the channel codewords are sufficiently long. The latter assumption is easily violated in a slowly varying fading channel with a delay-sensitive application that limits channel coding to a single realization of the channel. In this case, the end-to-end system should be designed using a joint source-channel coding approach. The joint source-channel coding problem for quasi-static fading channels has been studied for an asymptotically high signal to noise ratio (SNR) [5,36,39,62], and in the finite-SNR regime using numerical techniques [21,23,77,83].
The transmission strategy for a given application depends on the availability of the channel state information (CSI). For slowly fading channels, estimating the CSI at the receiver (CSIR) is relatively simple and incurs a negligible loss in the transmission rate. The joint source-channel coding literature cited in the preceding paragraph falls into the category of CSIR-only problems where the transmitter deals with the uncertainty of the channel using various layering schemes. On the other hand, the availability of a feedback link allows one to improve the performance using the CSI at the transmitter (CSIT). The perfect CSIT scenario is usually costly due to the limited feedback resources at the receiver and a compromise can be made by quantizing the CSIT. In the context of rate maximization, the noiseless and noisy quantized feedback problems have been studied in [52] and [15], respectively.

The joint source-channel coding problem with noiseless quantized feedback was studied in [21] from both low and high SNR perspectives. In this chapter, we consider a more realistic scenario where the feedback link is noisy. To incorporate the effects of feedback errors, we formulate the distortion minimization problem in the context of channel optimized scaler quantizer (COSQ) design. Compared to a system without feedback, the proposed COSQ provides a performance gain that increases as the feedback quality improves. In an asymptotically high SNR regime, we characterize the system performance in terms of the distortion exponent metric that is defined as the slope of the expected distortion with respect to the SNR. We analytically show that although the COSQ-based system is superior to a single-layer no-CSIT system, the distortion exponents of the two schemes are the same.
5.0.5 Organization and Notation

This chapter is organized as follows. In Section 5.1, we present the system model. We then consider the COSQ design problem in Section 5.2. The asymptotic analysis is provided in Section 5.3. Numerical results are presented in Section 5.4. Finally, Section 5.5 concludes this paper.

In the sequel, $\| \cdot \|$ is the Euclidean norm of a complex vector. $P(e)$ represents the probability of event $e$ and $P(e_1, e_2)$ represents the joint probability of events $e_1$ and $e_2$. $F(x)$ represents the cumulative distribution function (CDF) of a given random variable. $\log$ and $\ln$ represent base 2 and natural logarithms, respectively. Exponential equality $f(x) \doteq x^\beta$ denotes $\lim_{x \to \infty} \frac{\log f(x)}{\log x} = \beta$ and $\geq$, $\leq$ are defined similarly.

5.1 System Model

We assume that the source is discrete-time complex Gaussian with independent real and imaginary components of variance 0.5. We consider mean square error (MSE) distortion measure and the distortion-rate (D-R) function of the source is given as $D(R_s) = 2^{-R_s}$, where $R_s$ is the source coding rate in bits per source symbol. The source coder encodes a block of $N$ source symbols and sends it to the channel encoder. We further assume that $N$ is large enough so that the source can be considered ergodic. The design of the source coder is not considered in our work, and we assume the availability of a source coder capable of achieving the distortion-rate bound.

The discrete-time, baseband model of a slowly fading, single-input multiple-
output channel with $M_r$ receive antennas is given as

$$r_t = hs_t + n_t \quad t = 1, \cdots, T$$

(5.1)

where $h$, $n_t$ and $r_t$ are $M_r \times 1$ complex vectors and $t$ is the index of the channel use. The channel coefficient vector $h$ is assumed to be random but constant over a sequence of $T$ channel uses. The elements of $h$ are independent and identically distributed (i.i.d) according to some known distribution. $s_t$ and $r_t$ represent the transmitted symbol and the received vector, respectively, in the $t^{th}$ channel use and $n_t$ is a complex Gaussian noise vector with independent real and imaginary parts of zero mean and variance 0.5. We assume that each codeword is limited to a single fading block, and the block length $T$ is large enough such that reliable transmission at a rate equal to the mutual information of the channel is possible when $h$ is known.

The encoded block of $N$ source symbols is transmitted in one fading block. For a transmission rate of $R$ bits per channel use, the source coding rate and the associated distortion are $R_s = \frac{TR}{N} = bR$ and $D(bR)$, respectively, where $b \triangleq \frac{T}{N}$ is the bandwidth expansion factor. The bandwidth expansion factor represents the number of channel symbols transmitted per source symbol and quantifies the compression or expansion of the source with respect to the available transmission bandwidth.

For a transmit power constraint of $P$, the instantaneous mutual information of the channel for a given $h$ is given as [91]:

$$I = \log(1 + \gamma P) \quad \text{bits/channel use}$$

(5.2)

where $\gamma \triangleq \|h\|^2$. Throughout the chapter, we assume that the receiver has perfect knowledge of $h$ and employs maximum ratio combining. Perfect knowledge of
γ at the transmitter enables outage-free transmission and since the block length T is large, the source-channel separation theorem holds in this scenario and the minimum distortion can be achieved using a transmission rate of \( R = I \). This justifies the selection of γ as the parameter to be quantized and sent to the transmitter when a limited rate feedback channel is available from the receiver to the transmitter [52]. In what follows, we describe the components of the system under study.

### 5.1.1 Transmission Scheme

The system block diagram is shown in Figure 5.1. For simplicity, we have shown the case of a single-input single-output (SISO) channel. The channel realization \( \gamma \) is quantized using a \( K \)-level scaler quantizer defined by the encoder and decoder mappings \( \Gamma_e : \mathbb{R} \to \mathcal{J} \) and \( \Gamma_d : \mathcal{I} \to \{\gamma_i\} \), respectively. \( \mathcal{J} = \mathcal{I} = \{0, \cdots, K - 1\} \) represent the encoder and decoder index sets, respectively. The encoding operation is performed at the receiver. The Voronoi regions of the encoder are specified by
the partition boundaries \( \gamma_0^b = 0 < \gamma_1^b \leq \cdots \leq \gamma_{K-1}^b < \gamma_K^b = \infty \), and the encoder mapping is defined as \( \Gamma_e(\gamma) = j \), if \( \gamma \in [\gamma_j^b, \gamma_{j+1}^b) \). The encoder index \( j \in J \) is then sent to the transmitter using \( \log_2 K \) uses of a binary symmetric channel (BSC) with cross over probability \( \epsilon \). The probability of receiving index \( i \) at the transmitter given that index \( j \) was sent by the receiver is given as

\[
p_{ij} = \epsilon d(i,j) (1 - \epsilon)^{\log_2(K)-d(i,j)}
\]

where \( d(i,j) \) is the Hamming distance between the binary representations of \( i \) and \( j \). The quantizer decoding operation is performed at the transmitter as \( \Gamma_d(i) = \gamma_i \), where \( \{\gamma_i\} \) represents the COSQ codebook with \( \gamma_i \in [\gamma_i^b, \gamma_{i+1}^b) \). Upon receiving index \( i \), the transmitter allocates power \( P_i \) to a capacity achieving codeword and transmits at rate \( R_i = \log(1 + \gamma_i P_i) \). The transmission is then successful if \( \gamma \geq \gamma_i \), otherwise outage occurs. The source coding rate at the transmitter is \( bR_i \) and the corresponding distortion at the receiver is \( D(bR_i) \) if the transmission is successful. In the case of outage, the resulting distortion is \( D_0 = D(0) = 1 \). Note that unlike the layered transmission schemes \([5, 21, 23, 36, 39, 83]\), we do not require the source to be successively refinable.

### 5.1.2 Performance Measure

In this section, we derive the expected distortion that is the design criterion of the proposed transmission scheme. Transmission at rate \( R_i = \log(1 + \gamma_i P_i) \) is successful when the transmitter receives index \( i \), and \( \gamma \geq \gamma_i \). The probability of this event is \( \omega_i \triangleq P(\gamma \geq \gamma_i, i) \), and the associated distortion is \( D(bR_i) \). An outage
occurs with probability $P(\gamma < \gamma_i, i) = P(i) - \omega_i$. The expected distortion is then

$$E_D = \sum_{i=0}^{K-1} \left\{ \omega_i D(bR_i) + [P(i) - \omega_i] D_0 \right\}$$

$$= \sum_{i=0}^{K-1} \omega_i D(bR_i) + \left( 1 - \sum_{i=0}^{K-1} \omega_i \right) D_0$$  \hspace{1cm} (5.4)

Each $\omega_i$ can be written as

$$\omega_i = P(\gamma_i \leq \gamma < \gamma_{i+1}, i) + P(\gamma \geq \gamma_{i+1}, i)$$  \hspace{1cm} (5.5)

The two terms in (5.5) can be calculated as follows.

$$P(\gamma_i \leq \gamma < \gamma_{i+1}, i) = \sum_{j=i}^{K-1} p_{ij} \left[ F(\gamma_{b,j+1}) - F(\gamma_{b,j}) \right]$$

$$P(\gamma \geq \gamma_{i+1}, i) = \sum_{j=i+1}^{K-1} p_{ij} \left[ F(\gamma_{b,j+1}) - F(\gamma_{b,j}) \right]$$

As a result

$$\omega_i = p_{ij} \left[ F(\gamma_{b,j+1}) - F(\gamma_i) \right] + \sum_{j=i+1}^{K-1} p_{ij} \left[ F(\gamma_{b,j+1}) - F(\gamma_{b,j}) \right]$$  \hspace{1cm} (5.6)

### 5.1.3 Transmission Power Control

We consider two types of transmission power control strategies. Under the short-term power constraint, every codeword is allocated a power budget of $P_i = \mathcal{P}$, regardless of the transmitter feedback index $i$. The more relaxed long-term power constraint allows the transmitter to choose a transmit power $P_i$ for index $i$, while limiting the average transmit power to $\mathcal{P}$. The long-term power constraint is then expressed as

$$E_P = \sum_{i=0}^{K-1} v_i P_i \leq \mathcal{P}$$  \hspace{1cm} (5.7)

where

$$v_i \triangleq P(i) = p_{i0} F(\gamma_{b,i}) + \sum_{j=1}^{K-1} p_{ij} \left[ F(\gamma_{b,j+1}) - F(\gamma_{b,j}) \right]$$  \hspace{1cm} (5.8)
The long-term power constraint may result in practically unacceptable large values of $P_i$ for certain indices. In this case, one may also consider a maximum power constraint in the form of $P_i \leq P_m$.

### 5.2 Quantizer Design

Our goal is to design a system that could take advantage of feedback information even when the feedback channel is noisy. Moreover, the proposed scheme should perform close to a system without feedback when the feedback channel quality is poor, and its performance should improve for better feedback qualities. These goals can be met by designing a channel optimized scaler quantizer (COSQ) that incorporates the effects of unreliable feedback into the design. We first simplify this problem by reducing the optimization variables.

**Lemma 4.** For the noisy feedback system of Figure 5.1, the optimal system satisfies $\gamma_i^b = \gamma_i^*$ for $1 \leq i \leq K - 1$.

**Proof.** We use contradiction to prove the lemma. Consider the optimal quantizer, $Q^*$, with parameters $\{\gamma_i^*, \gamma_i^{b*}, P_i^*\}$ and assume that for some $j$, $\gamma_j^{b*} < \gamma_j^*$. Now define a second quantizer, $Q$, with the same parameters as $Q^*$ with an exception that $\gamma_j = \gamma_j^b = \gamma_j^{b*}$ for $Q$. Since $\epsilon$ is independent of $\gamma_j$, from (6.23) and (5.8) we conclude that the expected power, $E_P$, is not affected by this difference. Therefore, the expected power is the same for both quantizers and can be ignored when comparing $Q$ and $Q^*$. Now assume that $\gamma_j^{b*} < \gamma < \gamma_j^*$ and the receiver sends index $j$ while the transmitter receives index $i$. If $i < j$, the expected distortions of both quantizers is determined by $\gamma_i^*$. If $i > j$, both quantizers experience outage. In either case, the two quantizers are equivalent. However, if $i = j$ then $Q^*$ is in
outage while Q is not. This contradicts the optimality of $Q^*$. 

Using Lemma 4, $\omega_i$ and $v_i$ are simplified to

\[
\omega_i = \sum_{j=1}^{K-1} p_{ij} \left[ F(\gamma_{j+1}) - F(\gamma_j) \right] \tag{5.9}
\]

\[
v_i = p_{i0} F(\gamma_1) + \sum_{j=1}^{K-1} p_{ij} \left[ F(\gamma_{j+1}) - F(\gamma_j) \right] \tag{5.10}
\]

and the optimization problem is formally stated as

\[
\min_{\{\gamma_i, P_i\}} \mathcal{E}_D \tag{5.11}
\]

s.t. $\gamma_i \leq \gamma_{i+1}$, $P_i \geq 0$ and $\mathcal{E}_P \leq \mathcal{P}$

Optimization over $\{P_i\}$ is required only when long-term power control is employed. In this case, the optimization problem can be solved iteratively by minimizing over $\{\gamma_i\}$ for a given $\{P_i\}$, and vice versa. Since the expected distortion is non-negative and is reduced in each iteration, the algorithm is guaranteed to converge. Iterative optimization over a group of variables is known as the alternating optimization technique [27]. This algorithm provides an efficient, but not necessarily optimal, solution to problems with multiple variables where joint optimization over all variables is difficult.

We first consider the power optimization step in the case of long-term power control, assuming that $\{\gamma_i\}$ is given. The cost function $\mathcal{E}_D$ is subject to the power constraint of $\mathcal{E}_P \leq \mathcal{P}$. This constrained optimization problem can be solved using the Lagrange multiplier technique. Define the Lagrangian $\mathcal{L} = \mathcal{E}_D + \lambda \mathcal{E}_P$, where $\lambda \geq 0$ is the Lagrange multiplier. From (5.4), we note that the second term in $\mathcal{E}_D$ is a constant for a given $\{\gamma_i\}$ and can be eliminated from the cost function. Using (5.4) and (6.23), the optimization over $\{P_i\}$ is then equivalent to

\[
\min_{\{P_i\}} \sum_{i=0}^{K-1} \left\{ \omega_i D \left( b \log \left( 1 + \gamma_i P_i \right) \right) + \lambda v_i P_i \right\} \tag{5.12}
\]
Setting the derivatives $\frac{\partial C}{\partial P_i}$ to zero, we get

$$P_i = \left[ \frac{1}{\gamma_i} \left( \frac{b\omega_i\gamma_i}{\lambda\nu_i} \right)^{\frac{1}{1+b}} - \frac{1}{\gamma_i} \right]^+$$

where $[x]^+ = \max(x, 0)$. When the power allocation is additionally constrained by a maximum power constraint $P_i \leq P_m$, the water-filling operation is performed up to the maximum power, and the solution is given as

$$P_i = \left[ \min \left( \frac{1}{\gamma_i} \left( \frac{b\omega_i\gamma_i}{\lambda\nu_i} \right)^{\frac{1}{1+b}} - \frac{1}{\gamma_i}, P_m \right) \right]^+$$

To optimize over $\{\gamma_i\}$ when $\{P_i\}$ is given, we start with an initial solution corresponding to the noiseless case, and iteratively improve it. The noiseless solution can be found using the dynamic programming technique of [22]. The improvement step is done using the alternating optimization technique, where each variable $\gamma_i$ is alternatingly optimized over the interval $[\gamma_{i-1}, \gamma_{i+1}]$, while all other $\gamma_j$’s ($j \neq i$) are held constant. Since the quality of the solution depends on the starting point, the noiseless solution may not be a good candidate when the feedback channel is very noisy (large $\epsilon$). To address this issue, one can use the noiseless initial solution to design the noisy COSQ for some small $\epsilon$. The solution so found can then be used as the initial value for a new optimization with a larger $\epsilon$. This multi-step process can be repeated until the solution for the desired $\epsilon$ is obtained.

### 5.3 Asymptotic Analysis

In [22], we showed that single-layer transmission with $K$-level noiseless feedback and a short-term power constraint has the same distortion exponent as $K$-layer superposition coding without feedback [37]. The following theorem characterizes the distortion exponent when the feedback is noisy.
Theorem 5. The distortion exponent of the feedback scheme of Figure 5.1 is

$$\delta = \frac{bM_r}{b + M_r} \quad (5.15)$$

that is the same as the distortion exponent of a single-layer no-CSIT system \((K = 1)\).\(^1\)

Proof. First, consider the case of a short-term power constraint with \(P_0 = P_1 = \cdots = P_{K-1} = P\). We assume that in the high-SNR regime, \(R_i = r_i \log P\), where \(r_i\) is the spatial multiplexing gain of layer \(i\). We further assume that \(R_i \leq R_{i+1}\) for all \(i\). The corresponding outage probability is given as

$$F(\gamma_i) = P_{out}(r_i \log P) \triangleq P^{-d^*(r_i)} \quad (5.16)$$

where \(d^*(r) = M_r(1 - r)\) is the optimal diversity at the multiplexing gain of \(r\) for the SIMO system [103]. Since \(r_i \leq r_{i+1}\), \(d^*(r_i) \geq d^*(r_{i+1})\) and from (5.16) we have

$$\lim_{P \to \infty} [F(\gamma_{i+1}) - F(\gamma_i)] = \lim_{P \to \infty} F(\gamma_{i+1}) \quad (5.17)$$

where the last equality is due to fact that \(F(\gamma_j) = P^{-d^*(r_j)}, 0 \leq j \leq K - 1\), vanishes as \(P \to \infty\) and \(p_i|_{K-1}F(\gamma_K) = p_i|_{K-1}F(\infty) = p_i|_{K-1}\). Thus, \(\lim_{P \to \infty} \sum_{i=0}^{K-1} \omega_i = \sum_{i=0}^{K-1} p_i|_{K-1} = 1\). Since \(D(bR_i) = 2^{-bR_i}, \log P = P^{-br_i}\) the expected distortion in (5.4) can be expressed as

$$\mathcal{E}_D \triangleq \sum_{i=0}^{K-1} p_i|_{K-1}P^{-br_i} \geq \sum_{i=0}^{K-1} P^{-br_i} \quad (5.18)$$

\(^1\)Since the multiple transmit antenna channels cannot be considered as degraded, we only discussed the results with multiple receive antennas. However, if we accept sub-optimal transmission strategies, for multiple transmit antenna transmission, we can split the power among different spatial directions, similar to the fixed-rate system case. In this case the distortion exponent can be modified by multiplying \(M_t\) by the system dimension.
The distortion exponent is found by maximizing the minimum exponent in (5.18). Since \( r_i \leq r_{i+1} \), the solution to this max-min problem is \( r_0 = r_1 = \cdots = r_{K-1} \). Furthermore, \( R_i = \log(1+\gamma_i P) = r_i \log P \). As a result, \( \gamma_0 = \gamma_1 = \cdots = \gamma_{K-1} \), that means the optimal distortion exponent is achievable with \( K = 1 \), or equivalently, without any feedback.

Now consider a long-term power constraint. Using a derivation similar to (5.17), we can show that \( \lim_{P \to \infty} v_i = P_i K^{-1} = \lim_{P \to \infty} \omega_i \). Moreover, from (5.13) we have

\[
\frac{P_i}{P_{i+1}} = \frac{\gamma_{i+1}}{\gamma_i} \left( \frac{(b \gamma_i)^{b+1}}{\lambda^{b+1}} - \frac{(b \gamma_{i+1})^{b+1}}{\lambda^{b+1}} \right)
\]

As \( P \to \infty \), we approach an unconstrained problem and as a result, \( \lambda \to 0 \) and

\[
\frac{P_i}{P_{i+1}} = \left( \frac{\gamma_{i+1}}{\gamma_i} \right)^{\frac{b}{b+1}}
\]

Since \( \gamma_{i+1} > \gamma_i \), (5.20) indicates that \( P_{K-1} \leq P \) otherwise, \( P_i > P \) for all \( i \) and the power constraint will be violated. As a result,

\[
P_{K-1} \leq P
\]

Since \( v_i \) is a bounded, multiplicative factor, the asymptotic power constraint is

\[
\mathcal{E}_P \approx \sum_{i=0}^{K-1} v_i P_i \approx \sum_{i=0}^{K-1} P_i \leq P
\]

For \( i < K-1 \), we have \( P_i + P_{K-1} \leq P \), or \( P_i \leq P - P_{K-1} = P^\beta \) for some \( \beta \leq 1 \). The expected distortion is then lower bounded as

\[
\mathcal{E}_D \approx \sum_{i=0}^{K-1} P_i^{-br_i} \geq \sum_{i=0}^{K-2} P_i^{-\beta br_i} + P_{K-1}^{-br_{K-1}} \geq \sum_{i=0}^{K-1} P_i^{-br_i}
\]

In (5.23), the first inequality is due to \( P_i \leq P^\beta \) and the second inequality comes from (5.21) and \( \beta \leq 1 \). Since the expected distortion is lower-bounded, the solution to the max-min problem of the bound is an upper bound on the distortion.
exponent. This solution is given as \( r_0 = r_1 = \cdots = r_{K-1} \). Now from (5.20),

\[
\log P_i - \log P_{i+1} = \frac{b}{b+1} (\log \gamma_{i+1} - \log \gamma_i)
\]  

(5.24)

At the lower bound, \( r_i = r_{i+1} \). Therefore,

\[
r_i \log P_i - r_{i+1} \log P_{i+1} = R_i - R_{i+1} = \frac{r_i b}{b+1} (\log \gamma_{i+1} - \log \gamma_i) \geq 0
\]  

(5.25)

since \( \gamma_i \leq \gamma_{i+1} \). Therefore, \( R_i \geq R_{i+1} \). But by assumption, \( R_i \leq R_{i+1} \), and as a result, \( R_i = R_{i+1} \). From (5.25), we conclude that \( \gamma_0 = \gamma_1 = \cdots = \gamma_{K-1} \). Thus, we have shown that an upper bound on the distortion exponent can be achieved by merging all the reconstruction points into \( \gamma_0 \) and an allocated power of \( P_0 = P \). Therefore, the resulting distortion exponent is equivalent to that of a system without feedback.

\[ \square \]

The intuition behind Theorem 5 is the following. In the high-SNR regime, the outage events caused by the erroneous feedback become the bottleneck. To avoid index errors and outage, therefore, the Voronoi regions must merge to a single region as in the case of no feedback.

### 5.4 Numerical Results

In this section, we present the numerical results for a Rayleigh fading channel. For comparison, we have included the results for a no-feedback scenario with \( n \)-layer superposition coding [23,77] and the noiseless feedback results of [22].

Figure 5.2 shows the results for \( K = 8, \ b = 1 \) and a short-term power constraint. The quality of the feedback channel is determined by the BSC cross-over
Figure 5.2: Expected distortion for a $1 \times 1$ system with $K = 8$, $b = 1$ and a short-term power constraint probability $\epsilon$. We note that a feedback channel with $\epsilon = 0.002$ provides a performance close to the noiseless feedback up to the channel SNR of 30 dB. Figure 5.2 also shows that as the quality of the feedback channel degrades, the system performance converges to that of a system without feedback that does not employ superposition coding ($n = 1$). A no-CSIT system with 5-layer superposition coding, however, has a distortion exponent equal to that of a noiseless feedback scheme with $K = 5$ and therefore, outperforms a noisy feedback scheme at high SNR [22].

From a quantization perspective, Figure 5.3 shows how increasing $\epsilon$ shrinks the Voronoi regions into a single threshold corresponding to the no-feedback scenario. The fact that the COSQ always performs between the two extremes of no feedback and noiseless feedback demonstrates its ability to exploit the CSIT even when the
quality of the feedback channel is poor. We also observe from Figure 5.2 that in the high SNR regime, the slope of the curve for $\epsilon = 0.2$ is roughly the same as the slope of the single-layer no-CSIT system, as predicted by Theorem 5.

Figure 5.4 shows the results for a 1x2 SIMO channel with $K = 4$, $b = 1$ and a short-term power constraint. The general behavior of the expected distortion in Figure 5.4 is similar to the SISO case. Figure 5.5 shows the absolute gain, defined as the reduction in the expected distortion when temporal power adaptation (long-term) is used compared to when there is no such adaptation (short-term). The absolute gain is zero for a no-CSIT system since power adaptation is not possible. As the quality of feedback improves, power adaptation provides more gains and absolute gain increases. For a given noise level in the feedback channel, however, the absolute gain vanishes at high SNRs. In this regime, an index error event is
the main cause of outage, and as a result, the COSQ Voronoi regions merge to reduce the probability of such an event. Thus, the COSQ-based design converges to a no-CSIT system that has zero absolute gain.

5.5 Conclusions

In this chapter, the problem of transmitting a Gaussian source over a single-input multiple-output quasi-static fading channel with a delay constraint was considered. The goal was to reconstruct the source at the receiver with minimum distortion when a limited-rate noisy feedback channel is available from the receiver to the transmitter. We proposed a COSQ to minimize the effects of the errors in the feedback channel and showed that the performance of the COSQ-based system...
Figure 5.5: Absolute gain for a $1 \times 2$ system with $K = 4$ and $b = 1$

falls between the noiseless feedback and no feedback scenarios, depending on the quality of the feedback channel. Numerical results for a Rayleigh fading channel showed the effectiveness of the COSQ at finite SNRs. Finally, we analytically showed that while quantized feedback is very effective at finite SNRs, it has the same distortion exponent as a no-CSIT system at asymptotically high SNRs.
Chapter 6

Throughput Maximization Over Slowly Fading Channels Using Erroneous Quantized Feedback

In practical communications, delay requirements limit the transmission codewords to a finite number of fading blocks and block fading Gaussian channel models must be adopted [6]. We are interested in a slowly fading channel model where each codeword is transmitted over a single fading block. The conventional performance measure for this application is the expected rate versus the outage probability [8]. Acquiring channel state information at the transmitter (CSIT) requires a feedback link and the optimal transmission scheme depends on the availability of CSIT. At one extreme, when perfect CSIT is available, outage-free transmission is possible at a rate equal to the current realization of the channel mutual information. Moreover, the transmission rate can be further improved if temporal power control is employed at the transmitter [30]. On the other hand, when CSIT is not available,
the best transmission strategy is superposition (multi-layer) coding [9, 84, 92, 100]. Between these two extremes, one can consider a scenario where only a quantized version of the CSIT is available [15, 52, 55].

Using CSIT with no feedback errors reduces the transmitter’s uncertainty about the channel realization to a single quantization bin. To further improve the performance, one can employ superposition coding over each quantization bin [52]. The latter work, only considers a noiseless feedback link. As the number of feedback regions increases, the channel uncertainty is reduced and superposition coding becomes less effective. Note that it is preferable to reduce the feedback channel code rate by the aid of quantization and also avoiding channel coding. Feedback channels are severely limited in their power and bandwidth resources. Moreover, to reduce the delay incurred in the feedback stage of a closed-loop communications system, we intend to reduce the time duration of feedback transmission over the reverse link, when the transmitter needs to remain idle. In this work, we use the framework of [52] together with a more realistic noisy feedback link [15]. With this model, the CSI uncertainty is similar to a no-CSIT scenario, and moreover, we have an additional source of error in the system. We design a CSI quantizer using a structure similar to channel optimized scalar quantizers (COSQs) [24, 61]. In the design process, we define a bit-mapping scheme to represent feedback indices in terms of binary bits, named quasi-grey bit-mapping. We show that by designing proper COSQs, we can still outperform a no-CSIT system with any feedback error probability. In the presence of feedback channel errors, a COSQ structure, which is a joint source-channel coding module, can adjust the CSI source coding and feedback information channel coding rates at the feedback link for any given feedback channel bit rate and error probability. With low-quality feedback
in terms of its rate and error probability, temporal power control is less effective, but superposition coding provides significant gains and the latter gain increases as the feedback channel quality degrades. These properties result from the structure of the closed-loop system with COSQ that with highly erroneous feedback converges to a no-CSIT scheme, where temporal power adaptation is impossible, while superposition coding is optimal.

6.0.1 Organization and Notation

The rest of this chapter is organized as follows. In Section 6.1, we introduce the transmission scheme, the quantizer structure of the CSIT, and the feedback channel model. The discussion of power adaptation strategies used at the transmitter and the definition of the expected rates with superposition coding is presented in Section 6.2. In Section 6.3, we define the quantizer design problem and provide our solution techniques. We extend the problem in Section 6.4, assuming a finite-state
model of the fading channel on the reverse link. In Section 6.5, we present the numerical results and finally, Section 6.6 concludes the chapter.

In the following, $P(e)$ represents the probability of event $e$ and $P(e_1, e_2)$ represents the joint probability of events $e_1$ and $e_2$. Also, the cumulative distribution function (cdf) of random variable $X$ is represented by $F(x)$, where $x$ denotes the realization of $X$. Furthermore, all logarithms are natural unless stated otherwise.

6.1 System Model and Transmission Scheme

We consider transmission over degraded channels, such as single-input multiple-output (SIMO) fading channels. The corresponding problem for multiple-input systems is beyond the scope of this work [84]. Fig. 6.1 shows the block diagram of our transmission scheme. Over a multiple antenna slowly fading channel, the discrete-time, transmit/receive signal model can be represented as

$$y(t) = h x(t) + n(t) \quad : \quad t = 1, \cdots, T \quad (6.1)$$

In the above equation, $t$ represents the index of channel use over a fading block duration. Across a SIMO channel, the transmit signal $x(t)$ is a scalar and $h$ and $y(t)$ are the $1 \times M_r$ channel vector and the received signal in the $t^{th}$ channel use, respectively. The components of the channel vector $h$ are independent and identically distributed (i.i.d) according to some known distribution. Moreover, $n(t)$ is a circularly symmetric complex additive white Gaussian noise (AWGN) vector, with independent real and imaginary parts of zero mean and variance $1/2$. Each transmission codeword is limited to a single fading block and the block length $T$ is large enough such that transmission codewords with a rate equal to the instantaneous mutual information of the channel can be decoded successfully.
Throughout this chapter, we assume that the receiver has perfect knowledge of
the channel vector $h$. We define the magnitude of $h$ as its Frobenius norm square,
$\gamma \triangleq \|h\|^2_F$. Knowledge of $\gamma$ at the transmitter enables outage-free transmission
and achieving high data rates. Therefore, we choose $\gamma$ as the CSI parameter to be
quantized [52]. The variable $\gamma$ is quantized using a $K$-level scaler quantizer defined
by the encoder and decoder mappings $\Phi_e: \mathbb{R} \rightarrow \mathcal{J}$ and $\Phi_d: \mathcal{I} \rightarrow \{\gamma_i\}$, respectively.
Here, $\mathbb{R}$ is the field of real numbers and $\mathcal{J} = \mathcal{I} = \{0, 1, \cdots, K - 1\}$ represent
the encoder and decoder sets of indices for the quantized CSI, respectively. The
quantization operation is performed at the receiver. The Voronoi regions of the
quantizer’s encoder are specified by the partition boundaries $\gamma^0_0 = 0 < \gamma^0_1 \leq \cdots \leq \gamma^{K-1}_K < \infty$, and the quantizer mapping is defined as $\Phi_e(\gamma) = j$, if $\gamma \in [\gamma^0_j, \gamma^0_{j+1})$. The encoder index $j \in \mathcal{J}$ is then conveyed back to the transmitter
through the feedback link. Unlike the original model of [52], we allow noise in the
feedback channel and consequently, the index $i \in \mathcal{I}$ could be different from $j$.

Quantizer design for noisy feedback channels requires the knowledge of the
index transition probability $p(i|j)$, defined as the probability of receiving index
$i$ at the transmitter, given index $j$ was sent from the receiver. Throughout this
chapter, we model the noisy feedback link as a discrete memoryless channel (DMC)
based on $\log_2 K$ uses of binary symmetric channels (BSCs) carrying the feedback
bits, with the bit error (cross-over) probability $\rho_f$. Moreover, first, we assume
that $\rho_f$ is a constant, known a priori to the transmission ends. A more elaborate
finite-state feedback channel model is studied in the end of the chapter, along with
the effects of mismatches about the prior knowledge of $\rho_f$. For a given cross-over
probability $\rho_f$, the index transition probabilities are given as

$$p(i|j) = \rho_f^{d_{i,j}} (1 - \rho_f)^{-d_{i,j}}, \forall i, j$$  \hspace{1cm} (6.2)$$

where $d_{i,j}$ is the Hamming distance between the binary representations of $i$ and $j$.

Generally speaking, the structure of the optimal quantizer depends on the mapping of quantization indices to binary bits or the bit-mapping scheme, $j' = \delta(j)$ and $i = \delta^{-1}(i')$, used in the system. Our design strategy is based on assuming a given one-to-one bit-mapping scheme at the feedback link. To facilitate the analysis and design of the optimal quantizer, we assume a specific class of bit-mappings where the index transition probability matrix of the DMC at the feedback link has the following properties:

$$\forall \ell \neq j : \sum_{k=j}^{K-1} p(k|j) \geq \sum_{k=j}^{K-1} p(k|\ell)$$ \hspace{1cm} (6.3)$$

$$\forall \ell > j, \forall m \leq K - 1 : \sum_{k=m}^{K-1} p(k|\ell) \geq \sum_{k=m}^{K-1} p(k|j)$$ \hspace{1cm} (6.4)$$

We call such a mapping scheme with properties (6.3)-(6.4), a quasi-grey bit-mapping. We refer to the next chapter for some examples of quasi-grey bit-mappings with various quantizer codebook cardinalities $K$. For example, for $K = 4$, a quasi-grey bit-mapping can be realized as $[19]

reconstruction points $\rightarrow$ $P_0$ $P_1$ $P_2$ $P_3$

quasi-grey bit-mapping $\rightarrow$ 00 11 10 01

[19] for more examples.
6.2 Transmission Scheme, Expected Rate, and Expected Power with Superposition Coding

With no-CSIT, the optimal transmission strategy is multi-layer (superposition) coding with successive interference cancelation and decoding \([9,84]\). With noiseless quantized CSIT, the uncertainty about the channel realization remains at the transmitter, but it is reduced to a single Voronoi region. Multi-layer coding within each region improves the performance, but its effectiveness reduces as the feedback regions become smaller, or equivalently, the feedback rate increases \([52]\). With noisy quantized CSIT, however, the situation is different. Upon receiving an index at the transmitter, the uncertainty about the actual channel realization remains on the whole domain of \(\gamma \in [0, \infty)\).

For superposition coding with \(L\) code layers, the quantizer’s decoding operation is performed at the transmitter as a one-to-many mapping \(\Phi_d(i) = \{\gamma_{i\ell}\}, \ \ell = \{0, \cdots, L-1\}\). Upon receiving index \(i\), \(0 \leq i \leq K - 1\), the transmitter sends the sum of \(L\) Gaussian codewords with rates \([52,92]\)

\[
R_{i\ell} = \log \left(1 + \frac{\gamma_{i\ell} P_{i\ell}}{1 + \gamma_{i\ell} \sum_{\kappa=\ell+1}^{L-1} P_{i\kappa}}\right), \quad 0 \leq \ell \leq L - 1
\]

where \(\{\gamma_{i\ell}\}\) represents the reconstruction points and the allocated power to the \(\ell^{th}\) codeword is \(P_{i\ell}\) and \(\log(\cdot)\) represents natural logarithm. The total transmit power in quantization bin \(i\) is \(P_i = \sum_{\ell=0}^{L-1} P_{i\ell}\). The fundamental assumption in applying this coding scheme is \(\gamma_{i\ell} \geq \gamma_{i(t-1)}\). With noiseless CSIT, the uncertainty associated with index \(i\) is confined to \([\gamma_i^b, \gamma_i^{b+1})\) and consequently, it is further assumed that \(\gamma_{i\ell} \in [\gamma_i^b, \gamma_i^{b+1})\) for \(\ell = \{0, \cdots, L-2\}\) and \(\gamma_{iL-1}^b \geq \gamma_{i(L-1)}\) \([52]\). Note, however, that with noisy CSIT, the latter constraints are not necessary. The expanded regions of uncertainty lead us to relax these additional structural constraints. This relaxation
allows the reconstruction points of the lower quantization bins to overlap with the upper bins, such as $\gamma_{it} > \gamma_{i+1}$. Therefore, our quantizer is a generalization of the quantizer introduced in [52].

We assume that the index $i$ can be reliably communicated to the receiver, like a “genie”, with a negligible rate loss of $\lim_{T \to \infty} \frac{\log_2 K_T}{T}$ bits per channel use. Note that theoretically, the forward link can take indefinitely large number of channel uses. Therefore, communicating a limited number of bits to introduce the index $i$ to the receiver is always possible in parallel with the actual data. As a result, the receiver knows the rate of the transmitted codeword layers. Therefore, successive decoding of code layers $\ell = L - 1$ to $\ell = 0$ can be performed with sequentially subtracting each codeword from the received signal, if $\gamma \geq \gamma_{it}$. Considering the negligible rate loss for communicating $i$ to the receiver, the following sections will provide a tight upper bound on the actual reliable expected transmission rate.

Under a long-term (average) power constraint, we allow the transmitter to choose a transmission power $P_i = \sum_{\ell=0}^{L-1} P_{i\ell}$ for index $i$, while the average transmission power is constrained to $P$ that indicates the SNR. The expected transmission power can be written as

$$E_P = \sum_{i=0}^{K-1} P(i)P_i = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} p(i|j)P(j)P_i$$

$$= \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} p(i|j) \left[ F(\gamma_{i+1}^b) - F(\gamma_j^b) \right] \sum_{\ell=0}^{L-1} P_{i\ell}$$

(6.6)

where $F(\gamma)$ shows the cumulative distribution function (CDF) of the random variable $\gamma$. Therefore, the long-term power constraint can be expressed as $E_P \leq P$.

---

$^1$Communicating a limited number of bits to the receiver with negligible rate loss is only not possible if $\gamma \to 0$. This instance of channel realization would incur outage in the system under study, regardless of the knowledge of the receiver about the feedback index at the transmitter. Therefore, it does not violate the validity of the treatment of the genie information in our system model.
With superposition coding, transmission at rate $R_{i\ell}$ can be reliably decoded if index $i$ appears at the transmitter and $\gamma \geq \gamma_{i\ell}$. Therefore, the achievable expected rate is $E_R = \sum_{i=0}^{K-1} \sum_{\ell=0}^{L-1} P(\gamma \geq \gamma_{i\ell}, i) R_{i\ell}$, where $R_{i\ell}$ was defined in Equation (6.5). The decoding event probability $P(\gamma \geq \gamma_{i\ell}, i)$ depends on the location of the reconstruction point $\gamma_{i\ell}$ among quantization bins. This probability can be expressed as

$$P(\gamma \geq \gamma_{i\ell}, i) = \sum_{j=0}^{K-1} 1_j(\gamma_{i\ell}) \left[ P(\gamma_{i\ell} \leq \gamma \leq \gamma_{j+1}^b, i) + P(\gamma \geq \gamma_{j+1}^b, i) \right]$$

(6.7)

with the indicator function

$$1_j(\gamma_{i\ell}) = \begin{cases} 
1 & \text{if } \gamma_{i\ell} \in [\gamma_j^b, \gamma_{j+1}^b) \\
0 & \text{otherwise}
\end{cases}$$

(6.8)

Moreover, in (6.7), we have

$$P(\gamma_{i\ell} \leq \gamma \leq \gamma_{j+1}^b, i) = p(i|j) \left[ F(\gamma_{j+1}^b) - F(\gamma_{i\ell}) \right]$$

(6.9)

and

$$P(\gamma \geq \gamma_{j+1}^b, i) = \sum_{n=j+1}^{K-1} p(i|n) \left[ F(\gamma_{n+1}^b) - F(\gamma_n^b) \right]$$

(6.10)

Therefore, the achievable expected rate can be expressed as

$$E_R = \sum_{i=0}^{K-1} \sum_{\ell=0}^{L-1} \mu_{i\ell} R_{i\ell}$$

(6.11)

where we can use the definition of the rate reward $\mu_{i\ell}$ [52,92]:

$$\mu_{i\ell} = \sum_{j=0}^{K-1} 1_j(\gamma_{i\ell}) \left\{ p(i|j) \left[ F(\gamma_{j+1}^b) - F(\gamma_{i\ell}) \right] + \sum_{n=j+1}^{K-1} p(i|n) \left[ F(\gamma_{n+1}^b) - F(\gamma_n^b) \right] \right\}$$

(6.12)

### 6.3 COSQ Design with Superposition Coding

Quantizer design problem with superposition coding involves finding the values of the boundaries $\{\gamma_i^b\}$ and the reconstruction points $\{\gamma_{i\ell}\}$ of the quantizer, and the
power allocation factors \( \{ P_{i\ell} \} \) to maximize the expected rate. Using \( \mathcal{E}_R \) and \( \mathcal{E}_P \) formulations in (6.11) and (6.2), this problem can be expressed as

\[
\max \quad \mathcal{E}_R \\
\{ \gamma_{i\ell}, \gamma_{i0}, P_{i\ell} \} \quad i = 0, 1, \ldots, K - 1, \quad \ell = 0, 1, \ldots, L - 1 \tag{6.13}
\]

s.t.

\[
\gamma_{i0} \leq 0, \quad \gamma_{i\ell} - \gamma_{i(\ell+1)} \leq 0, \quad \mathcal{E}_P - \mathcal{P} \leq 0
\]

To proceed with the quantizer design, we adopt an iterative solution of the above multi-variable optimization problem [52]. First, to reduce the complexity of the solution algorithm, we add an additional constraint on the values of power levels assigned to different quantization bins, i.e., \( P_i = \sum_{\ell=0}^{L-1} P_{i\ell} \), and we assume that the sequence of bin power levels is nondecreasing:

\[
P_0 \leq P_1 \leq \cdots \leq P_{K-1} \tag{6.14}
\]

Then, in the following lemma, we show that this constraint can notably reduce the complexity of the solution algorithm.

**Lemma 5.** Using a quantizer with contiguous Voronoi regions and a quasi-grey bit-mapping scheme at the feedback channel, the optimal quantizer structure with superposition coding satisfies \( \gamma_{i0} = \gamma_{i0}^* \), for \( i = 1, 2, \ldots, K - 1 \). In other words, for those indices, each boundary value of the quantizer must coincide with the smallest reconstruction point of the corresponding bin.

**Proof.** The quantizer structure under study is the one explained in the system model of Section 6.1.1. Furthermore, a quasi-grey mapping scheme with properties (6.3)-(6.4) is employed at the feedback link. We use contradiction to prove the statement of the Lemma. Consider the two quantizer structures shown in Fig. 6.1. Suppose that the optimal quantizer structure resembles Quantizer A, where
for an arbitrary index \( j \in \{1, 2, \cdots, K - 1\} \), we have \( \gamma^b_j < \gamma_j^0 \), thus there is an interval \( \gamma \in \Psi_j = [\gamma^b_j, \gamma_j^0) \), with feedback index \( j \) at the receiver. We compare the performance of this system with the one using Quantizer B that possesses the same structure as Quantizer A, except that \( \gamma^b_j = \gamma_j^0 \) and the latter interval does not exist.

Using Quantizer A, for the channel realizations in \( \Psi_j \), the probability of outage or the probability that the transmission codeword does not contribute to the aggregate rate of the system can be expressed as \( P^A_{\text{out}} = \sum_{k=j}^{K-1} p(k|j) \). With Quantizer B, the same channel realizations experience the outage probability expressed as \( P^B_{\text{out}} = \sum_{k=j}^{K-1} p(k|j - 1) \). According to the property (6.3), we can confirm that \( P^A_{\text{out}} \geq P^B_{\text{out}} \).

Moreover, with Quantizer A, the average transmission power given \( \Psi_j \) occurs can be written as \( P^A_{\text{avg}} = \sum_{k=0}^{K-1} p(k|j) P_k \). With Quantizer B, the same channel realizations incur the average transmission power equal to \( P^B_{\text{avg}} = \sum_{k=0}^{K-1} p(k|j - 1) P_k \). Assuming that the power codebook \( \{P_i\} \) is non-decreasing as in (6.14), and noting the property (6.4) of the quasi-grey bit-mapping, we can confirm that \( P^A_{\text{avg}} \leq P^B_{\text{avg}} \).

The rest of the regions other than \( \Psi_j \) behave similarly with Quantizers A and B. Note that Quantizer B provides greater aggregate data rate compared to Quantizer A, with less power consumption. This contradicts the optimality of Quantizer A and the proof is complete. \( \square \)

Using Lemma 5, we can replace the boundary variables \( \{\gamma^b_i\} \) with the corresponding reconstruction points \( \gamma_i^0 \). As a result, the optimization variables for the expected rate cost function \( \mathcal{E}_R \) can be reduced to \( \{\gamma_{i\ell}, P_{i\ell}\} \) and the quantizer design problem can be simplified. The long-term power constraint and the expected
rate objective can be expressed as

$$E_P = \sum_{i=0}^{K-1} \left( p(i|0) F(\gamma_{i0}) + \sum_{j=1}^{K-1} p(i|j) \left[ F(\gamma_{(j+1)0}) - F(\gamma_{j0}) \right] \right) \sum_{\ell=0}^{L-1} \mathcal{P}_\ell \leq \mathcal{P} \tag{6.15}$$

$$E_R = \sum_{i=0}^{K-1} \sum_{\ell=0}^{L-1} \left( \sum_{j=0}^{K-1} R_{\ell j}(\gamma_{i\ell}) \cdot \left[ p(i|j) \left[ F(\gamma_{(j+1)0}) - F(\gamma_{j0}) \right] + \sum_{n=j+1}^{K-1} p(i|n) \left[ F(\gamma_{(n+1)0}) - F(\gamma_{n0}) \right] \right) \right) \tag{6.16}$$

To maximize $E_R$ in (6.16) subject to (6.15), we use the idea of alternating optimization [27] and we proceed the following algorithm:

1) For a given power allocation set, $\{\mathcal{P}_\ell\}$, find the reconstruction points $\{\gamma_{i\ell}\}$ for $0 \leq i \leq K - 1$ and $0 \leq \ell \leq L - 1$. We assume that the initial power allocation is uniform ($\mathcal{P}_\ell = \mathcal{P}/L$). The coupling between the quantization bins in (6.16) makes the optimization challenging. However, for a fixed set of $\{\gamma_{i0}\}$, $i = 1, 2, \cdots, K - 1$, the expected rate expression (6.16) decouples for $\gamma_{00}$ and $\gamma_{i\ell}$, $i = 0, 1, \cdots, K - 1$, $\ell = 1, 2, \cdots, L - 1$. We therefore use exhaustive search over the set of $\{\gamma_{i0}\}$, $i = 1, 2, \cdots, K - 1$. In each iteration of the search, we separately find the maximizers of the objective function with respect to $\gamma_{00}$ and $\gamma_{i\ell}$, $i = 0, 1, \cdots, K - 1$, $\ell = 1, 2, \cdots, L - 1$, by searching over $[\gamma_{i(l-1)}, \infty)$ in the $\ell = 1$ to $\ell = L - 1$ order. Note that as defined in the description of the quantizer structure in Section 6.1.1, the ordering of the variables, $\gamma_{i0} \geq \gamma_{(i-1)0}$, $i = 1, 2, \cdots, K - 1$ reduces the complexity of the search.

2) For a given set $\{\gamma_{i\ell}\}$, find the optimal $\{\mathcal{P}_\ell\}$ for $0 \leq i \leq K - 1$ and $0 \leq \ell \leq L - 1$. In this step, a multi-step numerical search can be used [52]. Choose a Lagrange multiplier $\lambda$ and invoke the solution of optimal power allocation over
parallel Gaussian channels from [92] to solve

$$\max \{ \mathcal{P}_{i\ell} \}$$

subject to

$$\sum_{i=0}^{K-1} \sum_{\ell=0}^{L-1} \mu_{i\ell} R_{i\ell}$$

$$= \mathcal{P}_{i\ell}$$

$$\sum_{i=0}^{K-1} \nu_i \left( \sum_{\ell=0}^{L-1} \mathcal{P}_{i\ell} \right) \leq \mathcal{P}$$

(6.17)

For each quantization region $i$ and given $\lambda$, find the optimal $\mathcal{P}_i$, sequentially in the order of $i = K - 1, K - 2, \cdots, 0$, using

$$\mathcal{P}_i = \max \left\{ \left[ \max_{\ell} \left( \frac{\mu_{i\ell}}{\nu_i \lambda} - \frac{1}{\gamma_{i\ell}} \right) \right]^+, \mathcal{P}_{i+1} \right\}$$

(6.18)

where $[x]^+ = \max(x, 0)$. Note that the above solution confines the space of valid quantization bin powers to satisfy (6.14). Next, for the given $\lambda$, define the utility function $\mathcal{U}(z)$ as

$$\mathcal{U}(z) = \arg \max_{\ell} \frac{\mu_{i\ell}}{\nu_i \lambda} + z - \lambda : z \in [0, \mathcal{P}_i]$$

(6.19)

The optimal power allocated to Layer $\ell$ in each region is the length of the interval defined by [52, 92]

$$\mathcal{P}_{i\ell}^* = \max_{z \in [0, \mathcal{P}_i]} z - \min_{z \in [0, \mathcal{P}_i]} z$$

subject to

$$\mathcal{U}(z) = \ell$$

(6.20)

Finally, in each step of the algorithm, for a given $\lambda$, if the consumed power from (6.15) is greater than the power constraint, increase $\lambda$ and if it is less than the power constraint, decrease $\lambda$ and reiterate until the power constraint is satisfied with equality.

3) Iterate between steps 1) and 2) until convergence. In each step, the expected rate cost function either increases or remains unchanged. As a result, the algorithm converges to a locally optimal solution, since the expected rate is bounded above.
6.4 Problem Formulation Using a Finite-State Model of the Feedback Channel

In this section, we explore the rate maximization problem of the previous section. However, our goal is to extend the study with a more elaborate model of the feedback channel. We define a finite-state feedback channel model based on the well-established model of [96]. To simplify the derivations, we are going to proceed based on a single-layer coding transmission strategy. The extension to the case of multi-layer coding is possible using the approaches taken in the previous sections. Therefore, in the sequel, we only discuss the parts of the quantizer design that is different from the derivations of the previous sections.

Here, we just consider single layer coding for simplicity and assume that upon receiving index $i$, the transmitter allocates power $P_i$ to a capacity achieving code-word and transmits at rate

$$R_i = \log(1 + \gamma_i P_i)$$  \hspace{1cm} (6.21)

The transmission is successful if $\gamma \geq \gamma_i$, otherwise outage occurs.

6.4.1 Finite-State Feedback Channel Model

In a wireless fading environment, it is reasonable to assume that the forward and feedback channels are reciprocal and the fading coefficients are the same in both directions [32]. The reciprocal channel provides a more accurate model for the feedback link than a noiseless channel, or a binary symmetric channel (BSC) with a fixed error rate for all channel realizations. Our proposed model for the feedback channel is a BSC with a bit error rate (cross-over probability) $\epsilon(\rho, \gamma)$ that depends
on the forward channel realization $\gamma$, and the power allocated to feedback signals at the receiver, $\rho$. For simplicity, we use an uncoded binary phase shift keying (BPSK) modulation scheme for the feedback signals. Since the receiver knows the channel coefficient $h$ perfectly, the feedback signal transmission scheme is a special case of beamforming. Therefore, the BPSK transmitted feedback signals are of form $\pm \sqrt{\rho} h^*/|h|$, with instantaneous power $\rho$. Here $h^*$ is the complex conjugate of $h$. The cross-over probability of the equivalent BSC is then $\epsilon(\rho, \gamma) = Q(\sqrt{2\rho\gamma})$.

The COSQ design procedure requires the index transition probabilities $p_{ij}$, defined as the probability of receiving index $i$ at the transmitter, given index $j$ was sent from the receiver. For simplicity, we assume that within a Voronoi region $j$, the BSC error rate is a constant $\epsilon_j = \epsilon(\rho, \gamma^d_j)$, for some $\gamma^d_j \in [\gamma^b_j, \gamma^b_{j+1})$. The latter assumption approximates the feedback link as a finite-state Markov channel in which each Voronoi region represents a state [96]. The selection of an appropriate value for $\gamma^d_j$ is addressed in the following parts. Assuming that the feedback information is transmitted at the rate of one bit per channel use, the index transition probabilities are given as

$$p_{ij} = (\epsilon_j)^{d_{i,j}} (1 - \epsilon_j)^{\log_2(K) - d_{i,j}}$$  \hspace{1cm} (6.22)

where $d_{i,j}$ is the Hamming distance between the binary representations of $i$ and $j$.

### 6.4.2 Transmission Power Control

Under a short-term power constraint, every codeword is allocated a power budget of $P_i = \mathcal{P}$, regardless of the transmitter feedback index $i$. The more relaxed long-term power constraint allows the transmitter to choose a transmit power $P_i$ for

---

$^2$The state transition probabilities of the finite-state model are irrelevant to our discussion since the coding does not extend across multiple channel states.
index $i$, while limiting the average transmit power to $P$. Therefore, for a noisy feedback channel, the expected transmit power can be derived as

$$
E_P = \sum_{i=0}^{K-1} P(i)P_i = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} p_{ij}P(j)P_i = \sum_{i=0}^{K-1} \sum_{j=0}^{K-1} p_{ij} \left[ F(\gamma_{j+1}^b) - F(\gamma_j^b) \right] P_i
$$

(6.23)

and the long-term power constraint is expressed as $E_P \leq P$. The long-term power constraint may result in practically unacceptable large values of $P_i$ for certain indices. In this case, one may also consider a maximum power constraint in the form of $P_i \leq P_m$.

### 6.4.3 COSQ Design with Finite-State Feedback Channel

Transmission at rate $R_i = \log(1 + \gamma_i P_i)$ is successful when the transmitter receives index $i$, and $\gamma \geq \gamma_i$. The probability of this event is

$$
P(\gamma \geq \gamma_i, i) = P(\gamma_i \leq \gamma < \gamma_{i+1}^b, i) + P(\gamma \geq \gamma_{i+1}^b, i)
$$

(6.24)

The two terms in (6.24) can be calculated as follows.

$$
P(\gamma_i \leq \gamma < \gamma_{i+1}^b, i) = P(i|\gamma_i \leq \gamma < \gamma_{i+1}^b)P(\gamma_i \leq \gamma < \gamma_{i+1}^b) = p_{ii} \left[ F(\gamma_{i+1}^b) - F(\gamma_i) \right]
$$

(6.25)

$$
P(\gamma \geq \gamma_{i+1}^b, i) = \sum_{j=i+1}^{K-1} p_{ij}P(\gamma_j^b \leq \gamma < \gamma_{j+1}^b) = \sum_{j=i+1}^{K-1} p_{ij} \left[ F(\gamma_{j+1}^b) - F(\gamma_j^b) \right]
$$

(6.26)

Using (6.24)-(6.26), the achievable expected rate, $E_R = \sum_{i=0}^{K-1} P(\gamma \geq \gamma_i, i) \log(1 + \gamma_i P_i)$ can be expressed as

$$
E_R = \sum_{i=0}^{K-1} \left\{ p_{ii} \left[ F(\gamma_{i+1}^b) - F(\gamma_i) \right] + \sum_{j=i+1}^{K-1} p_{ij} \left[ F(\gamma_{j+1}^b) - F(\gamma_j^b) \right] \right\} \log(1 + \gamma_i P_i)
$$

(6.27)

We will now consider the issue of selecting $\gamma_{d,j}$'s that determine the index transition probabilities. To make the rate maximization problem tractable, we choose $\gamma_{d,j}$ val-
ues that correspond to the worst-case scenario. The following lemma characterizes such a choice.

**Lemma 6.** (Worst-Case Design) For a given Voronoi region $j$, $E_R$ is an increasing function of $\gamma^d_j$. Moreover, a lower bound to the optimal expected rate, $E^*_R$, is achieved by setting $\gamma^d_0 = \gamma_0$, and $\gamma^d_j = \gamma^b_j$ for $1 \leq j \leq K - 1$.

**Proof.** We first show that an error in the feedback link can only reduce the expected rate. Let’s assume that an index $j$ was sent by the receiver, and an index $i \neq j$ is received by the transmitter. If $i < j$, the transmitted rate is $R_i < R_j$. If $i > j$, then outage occurs and the reliable transmission rate is zero. In either case, the transmitted rate is lower than the rate achievable by error-free feedback, $R_j$. As the error probability in the feedback link increases, the probability of the event $i \neq j$ increases and the expected rate decreases. As a result, $E_R$ is a decreasing function of $\epsilon_j$, or equivalently, an increasing function of $\gamma^d_j$. For $1 \leq j \leq K - 1$, $E_R$ is lower bounded by the value associated with the smallest $\gamma^d_j$, which is $\gamma^d_j = \gamma^b_j$.

From (6.24), we note that $E_R$ depends only on the $\gamma \geq \gamma_i$ event for all $i$, and as a result, the transition probability in the $[0, \gamma_0)$ does not affect the expected rate.

For the region $[\gamma_0, \gamma^*_i)$, $E_R$ is lower-bounded by the worst-case, that is $\gamma^d_0 = \gamma_0$. □

According to Lemma 6, the worst-case COSQ design problem is equivalent to solving a max-min optimization problem. In other words, the maximization of the expected rate is performed for the worst-case realization of the feedback channel. We note that quantizer design for the worst-case scenario results in COSQs that are robust against the design mismatch [45]. Another by-product of worst-case design is optimization simplification, as shown by the following lemma.

**Lemma 7.** (Variable Reduction) Assuming worst-case design, $\gamma^b_i = \gamma^*_i$ for $1 \leq
Proof. We prove the lemma by contradiction. Suppose that the maximum expected rate $E_R^*$ is achieved by $\{\gamma_i^*; \gamma_i^{b*}; P_i^*\}$. Assume that for some index $1 \leq j \leq K - 1$, $\gamma_j^{b*} < \gamma_j^*$. Now consider a new COSQ with parameters $\{\gamma_i^*; \gamma_i^{b'}\}$ where $\gamma_j' > \gamma_j^{b*}$, and $\gamma_i' = \gamma_i^{b*}$ for $i \neq j$. Since $\gamma_j' = \gamma_j' > \gamma_j^{b*}$, for the same power allocation $\{P_i^*\}$ Lemma 6 implies that a rate $E'_R > E_R^*$ can be achieved. This contradicts the optimality of $E_R^*$.

In what follows, we assume worst-case design and replace the boundary variables $\{\gamma_i^b\}$ with the centroids $\{\gamma_i\}$ for $1 \leq i \leq K - 1$. Since $\gamma_i^d = \gamma_j^d = \gamma_j$, the transition probability $p_{ij}$ becomes a function of $\gamma_j$. As a result, the optimization variables for the expected rate cost function $E_R$ reduce to $\{\gamma_i; P_i\}$.

Under the short-term power constraint $P_i = P$. Thus, the expected rate optimization problem can be stated as

$$
\max_{\{\gamma_i\}} E_R \quad : \quad i = 0, \cdots, K - 1
$$

s.t. $\quad \gamma_{i+1} - \gamma_i \geq 0$

(6.28)

Assuming that the Karush-Kuhn-Tucker (KKT) conditions hold, the optimization problem (6.28) was numerically solved in [52] for a noiseless feedback link. Due to the implicit dependence of $p_{ij}$ on $\gamma_j$ in (6.27), the KKT necessary conditions are difficult to establish when the feedback channel is noisy. We propose the following algorithm to efficiently optimize (6.28). We start with an initial solution corresponding to the noiseless case and iteratively improve it. The improvement step is done using the alternating optimization technique [27], where each variable $\gamma_i$ is alternatingly optimized over the interval $[\gamma_{i-1}, \gamma_{i+1}]$, while all other $\gamma_j$’s ($j \neq i$)
are held constant. Since $\mathcal{E}_R$ is bounded from the above and each iteration can only improve the cost function or leave it unchanged, the algorithm is guaranteed to converge.

As we pointed out in the proof of Lemma 6, the transition probability of the feedback channel in the $[0, \gamma_0)$ region does not affect the expected rate. The expected power, however, is affected by this region since the index received by the transmitter for such a realization nevertheless incurs some power consumption. The worst-case feedback assumption in this region is a BSC with a cross-over probability of 0.5. As a result, under the worst-case design assumption, the expected power given in (6.23) can be expressed as

$$E_P = \sum_{i=0}^{K-1} \nu_i P_i$$

(6.29)

where $\nu_i = p^0_i F(\gamma_0) + p^0_i [F(\gamma_i) - F(\gamma_0)] + \sum_{j=1}^{K-1} p_{ij} [F(\gamma_{j+1}) - F(\gamma_j)]$. Here, note that $p^0_{i0} = 1/K$. The rate maximization problem is now formally stated as

$$\max \quad \mathcal{E}_R \quad : \quad i = 0, \cdots, K-1$$

$$\{\gamma_i; P_i\}$$

s.t. $\gamma_{i+1} - \gamma_i \geq 0$, $\mathcal{E}_P \leq P$, $P_i \geq 0$

(6.30)

Similar to the technique proposed in [52], we solve this constrained optimization problem by introducing the Lagrange multiplier $\lambda$, and the Lagrangian $\mathcal{L} = \mathcal{E}_R - \lambda \mathcal{E}_P$. We then apply the alternating optimization principle to maximize $\mathcal{L}$ by alternating between the optimization over $\{\gamma_i\}$ given $\{P_i\}$, and vice versa. The starting point of the algorithm is the short-term power solution. Optimizing $\{\gamma_i\}$ when $\{P_i\}$ is given is also done through alternating optimization. This step implicitly satisfies the $\gamma_{i+1} \geq \gamma_i$ constraint. Note that any candidate $\{\gamma_i\}$ that violates the power constraint has to be skipped during the procedure. For a given $\{\gamma_i\}$,
the optimization over \( \{P_i\} \) is equivalent to
\[
\max_{\{P_i\}} \sum_{i=0}^{K-1} \left[ \omega_i \log(1 + \gamma_i P_i) - \lambda \nu_i P_i \right], \quad \text{where} \quad \omega_i = \sum_{j=i}^{K-1} p_{ij} \left[ F(\gamma_{j+1}) - F(\gamma_j) \right]
\]

The solution to the latter problem is readily available in the form of water-filling [9]
\[
P_i = \left[ \frac{\omega_i}{\lambda \nu_i} - \frac{1}{\gamma_i} \right]^+
\]
where \([x]^+ = \max(x, 0)\), and \(\lambda\) is chosen such that the power constraint is satisfied. When the power allocation is additionally constrained by a maximum power constraint \(P_i \leq P_m\), the water-filling operation is performed up to the maximum power, and the solution is given as
\[
P_i = \left[ \min \left( \frac{\omega_i}{\lambda \nu_i} - \frac{1}{\gamma_i}, P_m \right) \right]^+
\]

### 6.5 Numerical Results

In this section, we show the properties of the solutions to the rate maximization problem of the previous section. First in Fig. 6.2, we plot the reconstruction points of the optimized COSQs designed for a 1 \( \times \) 1 channel, using superposition coding with \( L = 2 \) code layers. The leftmost plot is the noiseless feedback quantizer and the rightmost one is the no-CSIT solution. As the quality of the feedback link degrades, the resulting quantizer converges to the no-CSIT solution. Moreover, for extremely noisy systems, the reconstruction points associated with different quantization bins overlap. For example, \( \gamma_{10} < \gamma_{01} \) for \( \rho_f = 0.25 \). Note that had we confined our system to non-overlapping regions (similar to [52]), the possibility of converging to a no-CSIT solution would have been diminished. Moreover, the proposed quantizer converges to a noiseless feedback solution for high-quality feedback links.
Figure 6.2: The reconstruction points of a $1 \times 1$ multi-layer coding system with $K = 2$-level quantizers and $L = 2$-layer codes at SNR = 40(dB) and with different BSC cross-over probabilities. The overlapping of the feedback regions is highlighted.

Fig. 6.3 shows the performance of the proposed scheme with a short-term (instantaneous) power constraint, where $P_i = \mathcal{P}$, $\forall i$. The forward link is a $1 \times 1$ channel and $K = 2$ feedback regions are used. In this figure, the performance of the system with $L = 1$ and $L = 2$ code layers and different feedback error values are shown. We can make the following observations. First, the performance of the system with noisy feedback falls between those of no-CSIT and error-free feedback systems. Second, the gain associated with superposition coding (layering gain) increases as the residual uncertainty (post feedback) increases, as in noisy feedback and the extreme case of no-feedback.

In Fig. 6.4, we investigate the properties of the solutions in more details. This figure shows the absolute gain of superposition coding versus the SNR. When a
short-term power constraint is used, the absolute gain is the same as the layering gain, i.e., the gain of superposition coding with respect to single-layer coding. When long-term power control is employed, however, the absolute gain has an additional component due to temporal power adaptation, namely, the power gain.

The following observations can be made from Fig. 6.4. First, as expected, the power gain diminishes with increasing the SNR, since the system becomes less power constrained. This can be seen from the convergence of short-term and long-term curves at high-SNR. Second, as the feedback quality degrades, the amount of layering gain increases. This result is due to the fact that for higher feedback error probabilities, the ambiguity regarding the real channel realization is increased and layering, as a technique for utilizing uncertainty, becomes more effective. Further-
Figure 6.4: Absolute gain of a $1 \times 1$ system with $L = 2$-layer superposition coding with short-term (SP) and long-term (LP) power constraints and $K = 2$ feedback regions.

more, at low-SNR, the absolute gain of high-quality feedback channels are larger. This can be explained noting that when power is scarce, power adaptation is more effective, provided that the feedback is reliable. In contrast, power adaptation loses its significance in the high-SNR regime and layering gain, which is more effective for systems with poor feedback, dominates the absolute gain. In summary, power gain is more effective for high-quality feedback links at low-SNR, while layering gain is more pronounced with poor feedback channels at high-SNR.

Next, we present the performance results of our proposed design scheme with finite-state feedback channel model. We emphasize that our proposed design strategy is based on the worst-case analysis. The numerical results of this section,
Figure 6.5: COSQ performance for a $1 \times 1$ Rayleigh fading channel with $K = 4$ (a) expected rate (b) Voronoi regions at $P = 20$ dB (c) absolute gains and the peak power constraint

however, are obtained using Monte Carlo simulations of one million channel realizations of $\gamma$. Each realization of $\gamma$ is encoded by the encoder of the designed COSQ and the resulting index is passed through a BSC with a cross-over probability $Q(\sqrt{2P\gamma})$. The COSQ decoder reconstructs $\gamma_i$ based on the index $i$ at the output of the BSC. The aggregate rate is $\log(1 + \gamma_i P_i)$ if $\gamma \geq \gamma_i$ and zero otherwise.

For two bits of feedback ($K = 4$) and a short-term power constraint, Fig. 6.5(a) presents the performance results for a $1 \times 1$ Rayleigh fading channel. The horizontal axis is the power constraint $P$ that represents the average SNR of the forward channel. The expected rates are shown for different values of the feedback signal power, $\rho$. It is seen from the figure that as we increase $\rho$, the feedback
quality improves, and the expected rate moves from a no-feedback solution towards a noiseless feedback solution. This behavior is also shown in Fig. 6.5(b) from a quantization perspective. The figure shows that as $\rho$ decreases, the Voronoi regions converge to a single region associated with the no-feedback scenario. The effects of employing long-term power constraint (temporal power control) are shown in Fig. 6.5(c). In this figure, the absolute gain is the performance improvement associated with using the long-term power constraint with respect to using the short-term constraint. Note that temporal power control is less effective for high transmit powers or low-quality feedback channels. Moreover, a peak power constraint $P_m = 1.25P$, reduces the expected rate at low and moderate transmit SNRs.

We will now investigate the effects of feedback modeling mismatch on the per-
formance of our scheme. The COSQ is designed for some feedback quality factor \( \rho \) and is evaluated using Monte Carlo simulations. We allow the value of \( \rho \) used in simulations to be different from the one used for COSQ design. Fig. 6.6(a) shows the simulation results for the noiseless feedback channel, as well as the one with \( \rho = -10 \) dB. The design and simulation values of \( \rho \) are shown by D and S, respectively. For example, “NL(D), -10 dB(S)” represents a COSQ that is designed using a noiseless feedback channel, and simulated at \( \rho = -10 \) dB. We see from the figure that a noiseless quantizer performs poorly when operating on a noisy feedback link. On the other hand, the performance of a COSQ designed for a noisy link improves only marginally when operating on a noiseless feedback link. This experiment clearly demonstrates the robustness of COSQ to design mismatch compared to a noiseless quantizer, under severe feedback link conditions.

The same experiment is repeated in Fig. 6.6(b) for \( \rho = 0 \) dB. Since the quality of the feedback link is improved, the COSQ and the noiseless quantizer perform almost identically at \( \rho = 0 \) dB. Finally, Fig. 6.6(c) presents the results of the same experiment for a 1 × 4 SIMO system at \( \rho = -10 \) dB. A pattern similar to the SISO case is observed for the SIMO scenario. The difference between the noiseless quantizer and the COSQ, however, is less pronounced, due to the diversity, and consequently, the higher quality of the feedback link in this case.

### 6.6 Conclusions

We considered the problem of maximizing the expected rate over a slowly fading channel when an unreliable and quantized version of the channel state information is available at the transmitter. The rate maximization problem was solved
with and without transmission power control protocols. Our proposed solution was based on a channel optimized scaler quantizer that is designed to incorporate noisy feedback in the transmission model. We proposed a quantizer structure for superposition coding that generalized the existing structure in the literature. Numerical results showed that for a high quality feedback channel, the proposed design performs close to the noiseless feedback, while its performance converges to the no-feedback scenario as the feedback channel quality degrades. We also demonstrated the robustness of our proposed solution against design mismatch in the feedback link model. More specifically, we showed that due to the additional uncertainty induced by the feedback errors, the multi-layer reconstruction points associated with a quantization bin may overlap with other bins. Our results also demonstrated that as the feedback quality degrades, the gain associated with temporal power adaptation (power gain) decreases. On the other hand, the gain associated with multi-layer coding (layering gain) increases for lower-quality feedback channels. These properties are attributed to the optimal quantizer structure of the system that converges to a no-feedback scheme with low-quality feedback information. We also assumed that the feedback channel suffers from fading, as well as a limited power budget and extended our design approaches to this case. Finally, through numerical analysis, we showed that the proposed scheme shows a desirable robustness against feedback channel modeling mismatch.
Chapter 7

Summary and Conclusions

In this dissertation, we discussed different wireless communication applications with limited and noisy channel state information at the transmitter (CSIT). The systems that we studied fall into two major categories as we review in the following sections. In all the proposed schemes, we follow a unified framework to design channel optimized quantizers and adjust the system parameters based on the feedback channel parameters.

7.1 Combining Space-Time Coding and Transmit Beamforming for Fixed Rate and Fixed Power Systems

For closed-loop multiple antenna coded modulation, the optimal transmission schemes depend on the availability of CSI at the transmitter. With a fixed power and fixed rate application, which was the focus of our work, the channel amplitude information was irrelevant. Because, the transmitter uses a certain power
budget that won’t be adapted based on the CSIT. Therefore, efficient quantization schemes at the receiver and the transmitter sides deal with the channel direction information. Conventionally, the transmission schemes designed in the literature work based on beamforming ideas. However, beamforming is not sufficient to achieve good performance results in several situations.

In this work, we showed that under two major conditions, we cannot apply beamforming directly and achieve desirable performance results. If i) the rate of the feedback information is very low compared to the dimension of the transmit antenna array or if ii) the feedback information is erroneous, conventional beamforming schemes fail to provide full diversity order. Therefore, we proposed several design schemes to combine space-time coding (STC) and beamforming (BF) to combat the CSI distortion due to the low-rate and erroneous quantized feedback.

We started with proposing a class of coded modulation schemes, named partly orthogonal designs (PODs) that provide a general framework for combining STC and transmit BF. In this context, we argued that if the cardinality of the feedback codebook is smaller than the number of transmit antennas, we have to combine STC and BF to achieve full transmit diversity order. The conventional way of combining was precoded space-time block codes (PSTBCs). PODs are more general structures compared to PSTBCs and therefore, in some situations in terms of the feedback rate and the system dimension, they can outperform PSTBCs in terms of the error performance. Moreover, they can be applied with less decoding and quantization complexities at the receiver. The attractive property of PODs is that their structure can be transformed closely to open-loop STCs with low-rate feedback and also to pure beamforming schemes when the feedback rate is high.

Then we used PODs in general with a new design of the precoder matrices to
deal with the issue of erroneous feedback. The design criterion in this context was defined based on directly minimizing the error rate of the system, instead of the conventional design criteria that maximize the received signal-to-noise ratio (SNR). We showed that the conventional design schemes cannot deal with noisy feedback suitably. The new design approach, however, can address low-rate feedback and erroneous feedback obstacles simultaneously.

The main goal of system design in combining coding and beamforming was to provide better system performance compared to open-loop STC schemes with any feedback conditions. Following the minimum error rate design criterion and our proposed precoder design technique, we were able to accomplish full-diversity performance even if the feedback is very low-rate and suffers from any error probability.

In the design process, two parts needed to be adjusted. One was the precoder design, which is related to the transmission scheme. The other one was the quantization scheme to represent the CSIT. Our approaches were based on the ideas of channel optimized vector quantizers (COVQs). Therefore, the design solutions also demonstrated the properties of the optimal COVQs in the literature. Namely, with high error probabilities at the feedback channels, the system converged to no-CSIT schemes, where the optimal transmission strategy is to spread the power across different spatial directions. With low-error probabilities and consequently error-free feedback, the system converged to the optimal directional beamforming schemes, where the transmitter projects all the power onto one spatial direction.

With combining STC and beamforming through precoder-based PODs, the system performance showed full-diversity properties with additional array gain compared to open-loop schemes. Therefore, the goals of combining where accom-
plished with our design approaches.

We also investigated the effects of design mismatch on the system performance. We argued that the design mismatch occurs when at the design stage, we make an assumption about the feedback channel error probability, where in reality, the error probability is different. This can be due to the uncertainties related to the attributions of the wireless feedback links. In this context, we proposed some design techniques based on worst-case assumptions and we showed that we can still accomplish the combining goals. That is even with erroneous feedback information with mismatched design assumptions we can provide full-diversity performance with additional array gains compared to open-loop schemes.

### 7.2 Rate and Power Allocation for Outage, Distortion, and Rate Optimization

In the second part of the work, we considered information theoretic aspects of wireless communications with limited and noisy CSIT. As discussed in the introduction, the most efficient closed-loop schemes in the literature studies the role of quantized channel magnitude information at the transmitter and designed variable power or variable rate transmission schemes when feedback channel is error-free. With available noiseless quantized CSIT, the uncertainty about the channel realization at the transmitter is limited to a single quantization bin. Therefore, for every bin, we can adjust the transmission power or rate to optimize the metric of interest. Feedback errors introduce an important source of error to the entire closed-loop communication system. Therefore, the design objective is to utilize feedback information even though through noisy feedback links and provide per-
formance improvements compared to a no-feedback scheme.

In the context of fixed-rate transmission, the task of the transmitter is to enable power control in the system and the appropriate design measure is the outage probability of the system. We showed that the optimal quantizer structure in such a scheme is a channel optimized scalar quantizer (COSQ) with a circular structure. The attractive property of a COSQ is that with highly erroneous feedback links, the optimal quantizer structure approaches a no-CSIT scheme. Therefore, the outage probability of the no-CSIT systems is the upper bound on the system outage probability with noisy feedback. We showed that the diversity gain of the system with noisy feedback is equal to the diversity gain of a no-CSIT system. Therefore, erroneous feedback cannot offer additional diversity gain, similar to noiseless feedback systems. However, the outage performance of the system is superior to a no-CSIT scheme with any feedback rate and error probability.

Moreover, we showed that power control is more effective than beamforming with fixed power when the amount of feedback resources are the same for the two systems. Furthermore, even noisy CSIT with power control can outperform an error-free feedback beamforming scheme. Therefore, we can claim that power control is more reliable and robust compared to beamforming.

To design the optimal quantizer structure with noisy feedback channel, we introduced a new scheme of mapping feedback indices to binary bits, called quasi-grey bit-mapping. Using a quasi-grey mapping, we could simplify the design of the optimal quantizer, which resulted in significant performance improvement compared to sub-optimal quantizer and mapping schemes.

Then, we extended the scope of the work to variable rate systems, where based on feedback information at the transmitter, we adjusted the rate and power of the
transmission codewords. Two objective functions were studied. In the context of joint source-channel coding with quantized feedback, we minimized the expected distortion of the system and showed that by optimizing the COSQ structure for quantizing CSI, the system outperforms a no-CSIT system. However, again the maximum diversity of the system is similar to the diversity of a no-CSIT scheme.

The major difference of fixed-rate and variable-rate systems is due to the requirement of receiver side information about the transmitter’s perception of the feedback information. Since, the transmitter changes the code rate based on feedback information, it is essential for the receiver side to know about the rate to be able to decode the transmitted codeword. Therefore, we suggested to use a flavor of genie channel for implementing receiver side information. To justify the existence of a genie channel, note that the amount of information that the genie channel must carry is equal to the feedback rate. However, in contrast to the short-range and time-sensitive feedback links, the genie channel in the forward link can bear the luxury of large delays. The receiver side information won’t be needed until the rest of the data frame is received, i.e., when the decoding starts. Communicating a fixed number of bits over the forward link with large duration becomes much more practical than reliable communication of feedback information in the reverse link. Therefore, genie idea in the forward link imposes much less restriction and complexity in terms of the practicality of system implementation, compared to the hypothetical noiseless feedback links.

In the context of rate maximization, we discussed more advanced coding schemes at the transmitter. As mentioned before, feedback information limits the uncertainty of the transmitter about channel realization. Therefore, we can employ superposition coding over the uncertain channel to achieve higher expected rates
compared to simple single-layer coding strategies. It was proven in the literature that additionally by increasing the feedback rate, the channel uncertainty vanishes at the transmitter and the gain of superposition coding diminishes. However, we showed that in a noisy feedback application, the uncertainty of channel realization is similar to a no-CSIT scheme and therefore, superposition coding is useful with any feedback rate. The gain of superposition coding (layering gain) is mostly pronounced at high-SNR regimes with more feedback error probabilities. Also we studied the behavior of the system when we employ power control at the transmitter. We showed that power control provides significant gains at low-SNR where the power is scarce and this gain is more realizable for more reliable feedback information. The system properties in terms of the layering gain and the power gain come from the structure of the optimal COSQ. With highly erroneous feedback, the quantizer structure approaches a no-CSIT system, where power control is impossible and superposition coding is optimal. On the other hand, with reliable feedback, the system approaches a noiseless feedback scheme, where power control demonstrates its maximum effects and superposition coding is useless.

We also introduced a finite-state feedback channel model, based on quantized feedback information, assuming that the fading channel is reciprocal. This is a step towards system design based on practical fading feedback channels. We showed that the same design principles as binary symmetric feedback channel can be applied with finite-state feedback links and the optimal quantizer structure and the system behaviors are similar in both cases.
7.3 Final Remark

All the above results show a similar property in terms of the optimal quantizer structures. We used the ideas of channel optimized quantizers to design the CSI quantizer structures across noisy feedback link. Channel optimized quantizers, COVQs and COSQs, are joint source-channel coding modules. Therefore, they provide a soft tradeoff between the source coding rate (CSI quantization rate) and the channel coding rate (the data protection level). Over a low-quality feedback channel, the optimal channel optimized quantizer degenerates to a system with less source coding rate, as some of the Voronoi regions of the optimal quantizer shrink. However, since the number of feedback bits are kept constant, the optimal COSQ spends the rest of the feedback bits for protecting the feedback data. This property of the COSQs is attractive in the sense that it can outperform all the conventional design approaches, where the quantizers are designed with less source coding rates and some channel coding schemes are used to protect the feedback bits. Therefore, the COSQ or COVQ structures proposed in this work can be utilized to improve the performance of the practical communication schemes that utilize CSIT through unavoidable noisy feedback channels.
Bibliography


