UCI EECS
EECS141B
Spring 2008

# HOMEWORK 3
**(Due 5/29/2008)**

1. Consider a second order PLL with the loop filter

$$G(s) = \frac{s+a}{s+b}$$

and assume that the transmitter phase receives an input

$$\phi(t) = (2\pi \Delta f t + \phi_0) u(t)$$

i.e., an abrupt change in the phase ($\phi_0$) and frequency ($\Delta f$) at time $t = 0$.

   (a) What is $\lim_{t\to\infty} e(t)$?

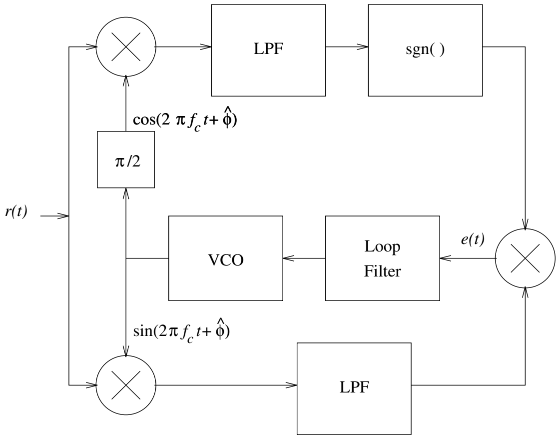
   (b) Recall that the transfer function for the VCO is modeled as $K/s$. Assume $K$ is given and finite. What are the conditions on $a$ and $b$ so that $\lim_{t\to\infty} e(t) = 0$?

   (c) Assume $a > 0$ and $b > 0$ are given. We wish to make $\lim_{t\to\infty} e(t) = 0$. Is there a way to accomplish this without changing $a$ or $b$?

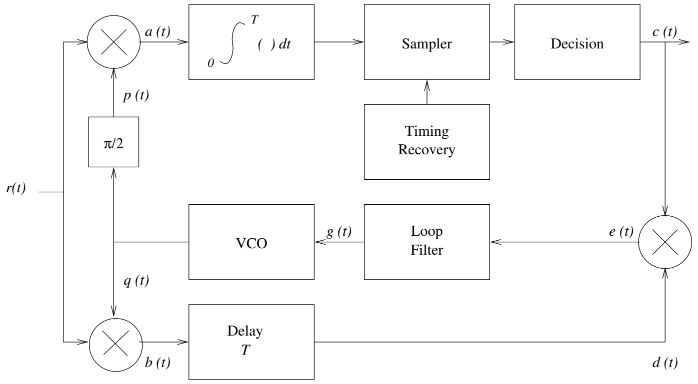
2. Consider the following Costas loop circuit for a BPSK signal. The transmitted signal is given as

$$s(t;\phi) = \cos(2\pi f_c t + \theta + \phi)$$

where $\phi$ is the unknown carrier offset and $\theta$ carries the information. The angle $\theta$ takes one of the values 0 or $\pi$. Generate a table listing values of $\theta$ and the values various points in the circuit take so that you calculate $e(t)$ for all possible values of $\theta$, and show that $\hat{\phi}$ indeed tracks $\phi$ via the circuit.



3.



The circuit above implements a combination of decision directed synchronization and Costas loop for BPSK. Assume, in the absence of noise, the received signal is in the form of $r(t) = A(t)\cos(2\pi f_c t + \phi)$ where $A(t)$ changes during each symbol period $(kT, (k+1)T]$, assuming values from $\{-A_c, A_c\}$. Assume $T = n/f_c$ where $n$ is a positive integer. Let

$$\begin{aligned} p(t) &= 2\cos(2\pi f_c t + \hat{\phi}) \\ q(t) &= 2\sin(2\pi f_c t + \hat{\phi}) \end{aligned}$$

where $\hat{\phi}$ is the receiver clock phase. The circuit is designed so that $\hat{\phi}$ can track the transmitter clock phase $\phi$ which may drift due to temperature changes, etc. Define $\Delta\phi = \phi - \hat{\phi}$.

(a) Why can't one use a simple PLL to extract the transmitter phase $\phi$?

(b) Calculate $a(t)$ and $b(t)$.

(c) Assuming $\Delta\phi$ is small, show that $c(t) = A(t)$ (ignore the delay due to the integrator).

(d) Considering that the upper and the lower rails have the same delay (through the integrator and the delay element, respectively), calculate $d(t)$ (i.e., ignore the effect of delay on both the upper and lower rails).

(e) Using your calculations of $c(t)$ and $d(t)$ above, calculate $e(t)$.

(f) Assuming the loop filter is low-pass, calculate $g(t)$.

(g) Can $g(t)$ be used to drive the VCO although the phase of $A(t)$ can change by 180°? Why?

4. The following are a set of classes for noiseless source codes.

   1. *Fixed-Length Codes:* A code whose codeword length is fixed.
   2. *Variable-Length Codes:* A code whose codeword length is not fixed.
   3. *Distinct Codes:* A code for which each codeword is distinguishable from all other codewords.
   4. *Prefix Codes:* A code in which no codeword is a prefix of another.

5. *Uniquely Decodable Codes:* A code for which the original source sequence can be reconstructed perfectly from the encoded binary sequence. All prefix codes are uniquely decodable but not all uniquely decodable codes are prefix codes.

Consider the following table where a source of size 4 has been encoded in binary codes with symbol 0 and 1.

| $s_i$ | Code 1 | Code 2 | Code 3 | Code 4 | Code 5 | Code 6 |
|---|---|---|---|---|---|---|
| $s_1$ | 00 | 00 | 0 | 0 | 0 | 1 |
| $s_2$ | 01 | 01 | 1 | 10 | 01 | 01 |
| $s_3$ | 00 | 10 | 00 | 110 | 011 | 001 |
| $s_4$ | 11 | 11 | 11 | 111 | 0111 | 0001 |

Fill in the following table by placing a check mark in the appropriate box if a code belongs to a particular class.

| Class | Code 1 | Code 2 | Code 3 | Code 4 | Code 5 | Code 6 |
|---|---|---|---|---|---|---|
| Fixed-Length | | | | | | |
| Variable-Length | | | | | | |
| Distinct | | | | | | |
| Prefix | | | | | | |
| Uniquely Decodable | | | | | | |

5. Define the entropy of a discrete source $\mathcal{S}$ with probabilities $p_k$, $k = 0, 1, \ldots, K-1$, as

$$H(\mathcal{S}) = \sum_{k=0}^{K-1} p_k \log_2\left(\frac{1}{p_k}\right) = -\sum_{k=0}^{K-1} p_k \log_2(p_k).$$

(a) Show that

$$H(\mathcal{S}) - \log_2 K = \log_2(e) \sum_{k=0}^{K-1} p_k \ln\left(\frac{1}{Kp_k}\right).$$

(b) Let $f(x) = \ln(x) - (x-1)$. By using calculus on $f(x)$ show that

$$\ln(x) \leq (x-1), \qquad x > 0$$

with equality at $x = 1$.

(c) Using the result in part (b) above in part (a), show that

$$H(\mathcal{S}) \leq \log_2(K)$$

with equality if and only if $p_k = 1/K$, $k = 0, 1, \ldots .K-1$.

6. Consider a discrete memoryless source with alphabet $\mathcal{S} = \{s_0, s_1, \ldots, s_{K-1}\}$ with probabilities $\{p_0, p_1, \ldots, p_{K-1}\}$, respectively. The $n$th extension of this source is another discrete memoryless source with source alphabet $\mathcal{S}^n = \{\sigma_0, \sigma_1, \ldots, \sigma_{M-1}\}$ where $M = K^n$.

Each $\sigma_j$ corresponds to a unique, ordered concatenation of symbols $s_{j_1}, s_{j_2}, \ldots, s_{j_n}$ from $\mathcal{S}$, $j = 0, 1, \ldots, M-1$, $0 \leq j_1, j_2, \ldots, j_n \leq K-1$. Note

$$P(\sigma_j) = P(s_{j_1})P(s_{j_2}) \cdots P(s_{j_n})$$

and

$$\sum_{j=0}^{M-1} P(\sigma_j) = \sum_{j_1=0}^{K-1} \sum_{j_2=0}^{K-1} \cdots \sum_{j_n=0}^{K-1} P(s_{j_1})P(s_{j_2}) \cdots P(s_{j_n}).$$

(a) Show that

$$\sum_{j=0}^{M-1} P(\sigma_j) = 1.$$

(b) Show that

$$\sum_{j=0}^{M-1} P(\sigma_j) \log_2 \left( \frac{1}{p_{j_k}} \right) = H(\mathcal{S}), \qquad k = 1, 2, \ldots, n.$$

(c) Hence, show that

$$H(\mathcal{S}^n) = \sum_{j=0}^{M-1} P(\sigma_j) \log_2 \left( \frac{1}{P(\sigma_j)} \right) = nH(\mathcal{S}).$$

(d) Calculate the maximum entropy of an ASCII symbol.

(e) Calculate the maximum entropy of an extended ASCII symbol.

(f) Do you think the entropy of an English text approaches that of the maximum entropy of an ASCII symbol? Why or why not?

7. Let a source alphabet $\mathcal{S}$ be specified in terms of its symbols and probabilities as in the following table.

| $s_k$ | $s_0$ | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ |
|---|---|---|---|---|---|---|
| $p_k$ | 0.30 | 0.24 | 0.21 | 0.14 | 0.07 | 0.04 |

(a) Calculate the entropy $H(\mathcal{S})$.

(b) Construct the Huffman code for $\mathcal{S}$. Tabulate the codeword corresponding to each symbol $s_k$.

(c) Calculate the average code length $\bar{L}$.

(d) Calculate the efficiency of the code, $\eta = H(\mathcal{S})/\bar{L}$.

8. Note

$$p(x_j, y_k) = p(x_j|y_k)p(y_k)$$

and

$$\sum_{j=0}^{J-1} p(x_j, y_k) = p(y_k).$$

Recall

$$\begin{aligned} H(\mathcal{X},\mathcal{Y}) &= -\sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(x_j,y_k)\log_2 p(x_j,y_k), \\ H(\mathcal{Y}) &= -\sum_{k=0}^{K-1} p(y_k)\log_2 p(y_k), \\ H(\mathcal{X}|\mathcal{Y}) &= -\sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(x_j,y_k)\log_2 p(x_j|y_k). \end{aligned}$$

Show that

$$H(\mathcal{X},\mathcal{Y}) = H(\mathcal{X}|\mathcal{Y}) + H(\mathcal{Y}).$$

9. (a) Show that

$$I(\mathcal{X};\mathcal{Y}) = H(\mathcal{X}) - H(\mathcal{X}|\mathcal{Y}) = \sum_{j=0}^{J-1}\sum_{k=0}^{K-1} p(x_j,y_k)\log_2 \frac{p(x_j|y_k)}{p(x_j)}.$$

(b) Using

$$\frac{p(x_j|y_k)}{p(x_j)} = \frac{p(y_k|x_j)}{p(y_k)}$$

and part (a) above, show that

$$I(\mathcal{X};\mathcal{Y}) = I(\mathcal{Y};\mathcal{X}).$$