Diversity Analysis of Single and Multiple Beamforming

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Abstract— **Multi-antenna communication systems have the potential to play an important role in the design of the next generation broadband wireless communication systems. In this paper, we study a single-user multi-antenna system with perfect channel state information (CSI) both at the transmitter and the receiver. Beamforming is used to exploit the perfect channel knowledge at both ends. We show that beamforming achieves the maximum diversity in space when only the best eigenmode is used (i.e., single beamforming). We extend our analytical results to multiple beamforming (i.e., sending more than one symbol simultaneously). Our main contribution is the analysis of the maximum achievable diversity order of beamforming systems.**

I. Introduction

In recent years deploying multiple transmit and receive antennas has become an important tool to improve the capacity and robustness of wireless communication systems [1]. In order to combat the severe conditions of wireless channels, the wireless systems should achieve a high diversity order. Many diversity techniques, some of which are temporal, frequency, code, and spatial have been developed.

Multi-input multi-output (MIMO) systems allow significant diversity gains for wireless communications. MIMO systems incorporating diversity generally can be grouped into two. The first group requires the channel state information (CSI) at the receiver, but not at the transmitter. Space-time (ST) codes are a subset of these systems and some important results can be listed as [2], [3], [4], [5]. The second group requires perfect or partial CSI at both the transmitter and the receiver. When perfect CSI is available at both ends, beamforming is used to maximize the SNR at the receiver. Beamforming separates the MIMO channel into parallel independent subchannels. When the subchannel with the largest gain is used for transmission, the technique is called single beamforming [6].

MIMO systems can also be used to enhance the throughput of wireless systems [7]. In the context of feedback systems, more than one subchannel can be used to improve the capacity. This technique is called multiple beamforming [6]. In this study we especially focus on the performance analysis of single and multiple beamforming with perfect CSI both at the transmitter and receiver. First, we show that single beamforming achieves maximum diversity available in space. We then calculate the maximum achievable diversity order for multiple beamforming.

The rest of the paper is organized as follows. Section II gives a brief overview of beamforming systems. The pairwise error probability analysis of both single and multiple beamforming are given in Section III. Simulation results supporting our analytical analysis are given in Section IV. Finally, we end the paper with a brief conlusion in Section V.

II. Beamforming Overview

If CSI is available both at the transmitter and the receiver, MIMO systems can benefit from significant diversity and coding gains by using beamforming. For frequency-division duplexing (FDD) systems, CSI can be obtained at the transmitter by using a feedback channel. One restriction is that the delay introduced by the feedback channel must be shorter than the coherence time of the wireless channel. However, for time-division duplexing (TDD) systems, there is no need for a feedback channel unless the channel is fast fading. Note that, to calculate CSI, the reverse channel should be excited prior to transmission on the forward channel.

Beamforming is implemented by multiplying the symbol(s) with appropriate beamforming vector(s) both at the transmitter and the receiver. In this paper, we assume that CSI is available at both ends. In such a case, the beamforming vectors are obtained via singular value decomposition (SVD) of the channel. Let's denote the quasi-static Rayleigh flat fading $N \times M$ MIMO channel as H, where N is the number of transmit antennas and M is the number of receive antennas. Then, the SVD of H can be written as

$$H = U\Lambda V^H = [u_1 u_2 \cdots u_N]\Lambda \left[\begin{array}{c} v_1 v_2 \ldots v_M \end{array} \right]^H \quad (1)$$

where Λ is a $N \times M$ matrix with singular values, $\{\lambda_i\}_{i=1}^{min(N,M)}$, in decreasing order on the main diagonal. U and V are two unitary matrices of size $N \times N$ and $M \times M$, respectively. By using SVD, MIMO channel is divided into independent and parallel subchannels.

A. Single Beamforming

Only one symbol is transmitted over the subchannel with the largest gain. The optimal vectors to be used at the transmitter side and receiver side are the first columns of U and V corresponding to the largest singular value of H. Then, the received signal can be represented by

$$y = xu_1^H H v_1 + \eta v_1 = \lambda_1 x + n \tag{2}$$

where λ_1 is the largest singular value of H, x is the transmitted symbol, $n = \eta v_1$, and η is complex additive white Gaussian noise (AWGN) vector of size $1 \times M$ with zero mean and variance $N_0 = 1/SNR$. The elements of H are modeled as complex Gaussian random variables with zero mean and 0.5 variance per complex dimension. Note that, the average total transmit power at the transmitter is assumed to be 1. Therefore, the received signal-to-noise ratio is SNR with the given channel and noise models.

B. Multiple Beamforming

Multiple symbols are simultaneously sent over different parallel independent subchannels with equal power allocation. If L subchannels are used, the input-output relation for the k^{th} subchannel becomes

$$y_k = \frac{1}{\sqrt{L}} \lambda_k x_k + n_k \tag{3}$$

where λ_k is the k^{th} largest singular value of H and $n_k = \eta v_k$.

III. Diversity Performance

A. Single Beamforming

In this section, by analyzing the pairwise error probability (PEP), we will show that single beamforming achieves the diversity order of NM for arbitrary N and M. References [8] and [9] emphasize the same result but do not give a formal proof for an arbitrary (N, M) pair. We will present an upper bound for PEP.

Assume that the symbol x is sent and $\hat{x}$ is detected. Then, using the maximum likelihood (ML) criterion, the PEP of x and $\hat{x}$ given CSI can be written as

$$\begin{aligned} P(x \to \hat{x}|\mathbf{H}) &= P\left(|y - \lambda_1 x|^2 \geq |y - \lambda_1 \hat{x}|^2\right) \\ &= P\left(\beta - |\lambda_1|^2 |x - \hat{x}|^2 \geq 0\right) \ \leq Q\left(\sqrt{\frac{|\lambda_1|^2 d_{min}^2}{2N_o}}\right) \end{aligned} \tag{4}$$

where $\beta = \lambda_1(\hat{x} - x)n^* + \lambda_1{}^*(\hat{x} - x)^* n$, d_{min} is the minimum Euclidean distance in the constellation, and $Q(\cdot)$ is the well-known Q-function. For given H, β is an independent zero-mean complex Gaussian random variable with variance $2N_o|\lambda_1|^2|x - \hat{x}|^2$. The inequality in (4) comes from the fact that $|x - \hat{x}| \geq d_{min}$. Using an upper bound for the Q function $Q(x) \leq (1/2)e^{-x^2/2}$, PEP can be bounded as

$$P(x \to \hat{x}) = E\left[P(x \to \hat{x}|\mathbf{H})\right] \leq E\left[\frac{1}{2}\exp\left(-\frac{|\lambda_1|^2 d_{min}^2}{4N_0}\right)\right] \tag{5}$$

Without loss of generality, we assume $N \leq M$. Since λ_1 is the maximum singular value, then

$$|\lambda_1|^2 \geq \frac{|\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_N|^2}{N} \ . \tag{6}$$

PEP can be given by

$$P(x \to \hat{x}) \leq E\left[\frac{1}{2}\exp\left(-\frac{d_{min}^2}{4N_0 N}\sum_{i=1}^{N}|\lambda_i|^2\right)\right] \tag{7}$$

Let's denote $|\lambda_i|^2 = \mu_i$. Note that $\{\mu_1, \mu_2 \ldots, \mu_N\}$ are the eigenvalues of HH^H in decreasing order [10]. The diversity order of single beamforming can be calculated using the joint pdf of μ_i's, which is given by [11],

$$\rho(\mu_1, \ldots, \mu_N) = K_{N,M}\prod_{i=1}^{N}\mu_i^{M-N}\prod_{i \geq j}(\mu_i - \mu_j)^2 e^{-\sum_{i=1}^{N}\mu_i} \tag{8}$$

where $K_{N,M}$ is a normalization constant that depends on both N and M. Using (7) and (8), PEP is upper bounded by

$$\begin{aligned} P(x \to \hat{x}) \leq & \int_{\mu_N} \ldots \int_{\mu_1} \frac{1}{2} K_{N,M} \prod_{i=1}^{N}\mu_i^{M-N}\prod_{i \geq j}(\mu_i - \mu_j)^2 \\ & \times e^{-\sum_{i=1}^{N}\mu_i\left(\frac{d_{min}^2}{4N_o N}+1\right)} d\mu_1 \ldots d\mu_N \end{aligned} \tag{9}$$

By simply making a change of variable, $\mu_i\left(\frac{d_{min}^2}{4N_o N} + 1\right) \to \mu_i$, it is easy to show that PEP is bounded by

$$P(x \to \hat{x}) \leq \frac{1}{2}\left(\frac{d_{min}^2}{4N_o N} + 1\right)^{-NM} \approx \frac{1}{2}\left(\frac{d_{min}^2}{4N}SNR\right)^{-NM} \tag{10}$$

for high SNR. From (10), it is easy to see that the diversity order of single beamforming is NM. It is straightforward to obtain the same result for $N > M$: all Ns should be replaced by M and all Ms should be replaced by N.

B. Multiple Beamforming

In the following section, using a similar analysis to that in the previous section, we will show that multiple beamforming achieves the diversity order of $(M - L + 1)(N - L + 1)$ for arbitrary N, M and L. Wihout loss of generality, we assume $N \leq M$. In multiple beamforming, multiple parallel subchannels are used for transmission of multiple symbols. However the performance is dominated by the weakest subchannel [12]. Therefore, when L symbols are transmitted, the PEP can be bounded by

$$\begin{aligned} P(x \to \hat{x}) &\leq E\left[\frac{1}{2}\exp\left(-\frac{\mu_L d_{min}^2}{4LN_0}\right)\right] \\ &= \int_{0}^{\infty} \frac{1}{2} e^{-\frac{\mu_L d_{min}^2}{4LN_0}} \rho(\mu_L) d\mu_L \\ &= G(\infty) - G(0) \end{aligned} \tag{11}$$

where μ_L is the L^{th} largest eigenvalue of HH^H, i.e., $\mu_L = \lambda_L^2$, and $\rho(\mu_L)$ is the corresponding pdf. To calculate the diversity order, the marginal pdfs of eigenvalues should be known. To the authors' knowledge, there is no explicit expression for marginal pdfs for an arbitrary (N, M) pair in the literature. However, for systems with a small number of antennas at the receiver and transmitter, one can find marginal pdfs using the joint pdf of ordered eigenvalues in (8). Then, we can analytically calculate the bounds for the PEP and diversity orders. To illustrate, we will give an example of how the diversity order of multiple beamforming is calculated.

Example: *Diversity order of 2 x 2 system (2 subchannels used).* The input-output relation for each subchannel is given by (3), where L is 2. The diversity order of the strongest (first) subchannel is 4 (proved in Section III-A). The diversity order of the second subchannel can be found using (11), where expectation is with respect to the pdf of the second largest eigenvalue. The pdf of μ_2 can be analytically found from (8) and can be expressed as,

$$\rho(\mu_2) = 2e^{-2\mu_2} \tag{12}$$

Using (11) and (12), PEP at high SNR can be bounded by

$$P(x \rightarrow \hat{x}) \leq \left(\frac{d_{min}^2}{8} SNR\right)^{-1} \tag{13}$$

As seen from (13), the diversity order of the second (weakest) subchannel is 1. For values of $\min(N, M)$ more than 4, expressions can be found for both marginal pdf of eigenvalues and PEP. However, the computational complexity to evaluate the formulas increases exponentially.

We will use an approximation to the marginal pdfs to achieve analytical results for arbitrary N, M and L. In (11), the resultant PEP is highly dependent on the values of μ_L around zero. The term $G(\infty)$ in (11) will be zero for large μ_L, since it has an exponential factor with negative exponent. Therefore, the pdf of L^{th} largest eigenvalue around zero is essential in determining the diversity performance [13]. For a better understanding, we will rephrase the approximation made for the i^{th} smallest eigenvalue in [13].

Approximation to the i^{th} smallest eigenvalue: Let ν_i and $\rho(\nu_i)$ be the i^{th} smallest eigenvalue and its pdf, respectively. By integrating (8) $N-1$ times, $\rho(\nu_i)$ of can be expressed as

$$\begin{aligned}\rho(\nu_i) = \int_0^{\infty}\int_0^{\nu_N} \cdots \int_0^{\nu_{i+2}}\int_0^{\nu_i} \cdots \int_0^{\nu_2} \rho(\nu_N, \ldots, \nu_1) \\ \times\, d\nu_1 \cdots d\nu_{i-1} d\nu_{i+1} \cdots d\nu_N \end{aligned} \tag{14}$$

where $\rho(\nu_N, \ldots, \nu_1)$ is the joint pdf of eigenvalues with ordering $\nu_N > \nu_{N-1} > \ldots > \nu_1$. Since our main concern is the marginal pdfs around zero, (14) can be further simplified assuming that ν_i is close to the origin. For every j and k smaller than i, ν_j and ν_k will also be close to the origin. Correspondingly, $(\nu_j - \nu_k)^2$ can be approximated as ν_k^2 for $j < k < i$ and $j < i < k$. With these assumptions, the integrals from $i-1$ to N in (14) can be separately calculated and result in a constant. The integrals over ν_k's, $1 \leq k < i$ can be calculated using the fact that $\int_0^x y^n e^{-y} dy \approx x^{n+1}/(n+1)$ for small x. The final form of $\rho(\nu_i)$ can be written as

$$\rho(\nu_i) = \nu_i^k h(\nu_i) e^{-\nu_i} \tag{15}$$

where $k = (M-N)+(i-1)(M-N+1)+2(1+2+\cdots+(i-1))$ and $h(\nu_i)$ is a function of ν_i consisting of polynomials and exponential functions of ν_i such that $h(0) \neq 0$. Let's define $F(\nu_i)$ as the probability distribution function of ν_i. From (15), it is easy to see that the first k derivatives of $F(\nu_i)$ evaluated at $\nu_i = 0$ are zero. Therefore, neglecting the higher order terms, the Taylor expansion of $F(\nu_i)$ around the origin can be approximated as

$$F(\nu_i) \approx F(0) + \alpha \nu_i^{k+1} \tag{16}$$

where $k+1 = i(M-N+i)$. To find the pdf for the i^{th} largest eigenvalue, just a change of variable i with $N-i+1$ is needed. Then, the pdf for the L^{th} largest eigenvalue can be approximated as

$$\rho(\mu_L) = \beta \mu_L^{(M-L+1)(N-L+1)-1} \tag{17}$$

where β is a constant. When (17) is used in (11), the PEP for multiple beamforming can be written as

$$\begin{aligned}P(x \rightarrow \hat{x}) &\leq \gamma \left(\frac{d_{min}^2}{4LN_oN}\right)^{-(M-L+1)(N-L+1)} \\ &= \gamma \left(\frac{d_{min}^2}{4LN} SNR\right)^{-(M-L+1)(N-L+1)}\end{aligned} \tag{18}$$

where γ is a constant. Consequently, as seen from (18), the diversity order for multiple beamforming is $(M-L+1)(N-L+1)$ when L subchannels are simultaneously used.

IV. Simulation Results

In this section, we provide simulation results that quantify some of the analytical results derived in this paper. Figure 1 shows the simulation results for different number of antennas with different substreams. As seen in Figure 1, first four SNR curves have the same diversity order of 4. The diversity order of 3×3 with one substream and 4×4 with two substreams are both 9. Also, the diversity order of 4×4 with one substream and 6×6 with three substreams are both 16 . Simulation results also verify that the diversity order of multiple beamforming is $(M-L+1)(N-L+1)$.

V. Conclusion

In this paper, our focus was on MIMO systems with full channel state information both at the transmitter and receiver. Our analysis was mainly based on the pairwise error probability and marginal pdf of channel eigenvalues. First, we showed that single beamforming achieves the maximum diversity order of NM in space over Rayleigh fading channels. Afterwards, we extended our analytical results for single beamforming to the multiple beamforming case. We analytically showed that the maximum achievable diversity order of multiple beamforming is $(M-L+1)(N-L+1)$. We provided the simulation results that agree with our analytical results.

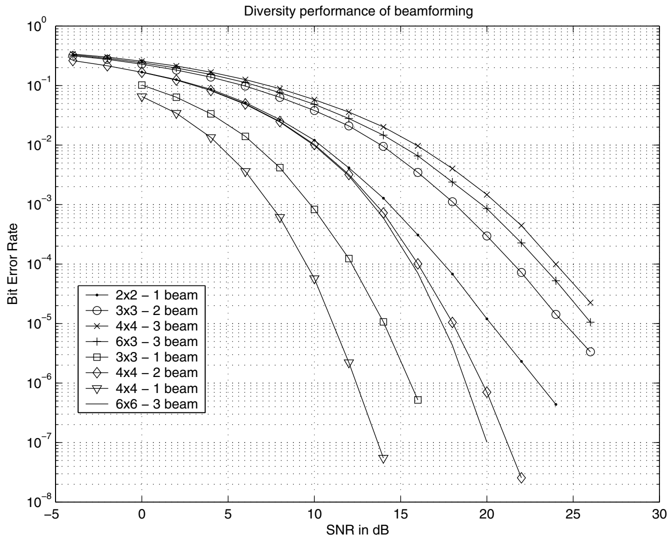

Fig. 1. Performance of beamforming with different antenna configurations for flat fading channels.

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