Network Coding for Three Unicast Sessions: Interference Alignment Approaches

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Outline

- Motivation
  - Network Alignment (NA: NC+IA)
- Network Alignment Approaches
  - at the edge
  - in the middle
- When is NA necessary?
  - Can other schemes achieve >= $\frac{1}{2}$ rate or more?
- Conclusion

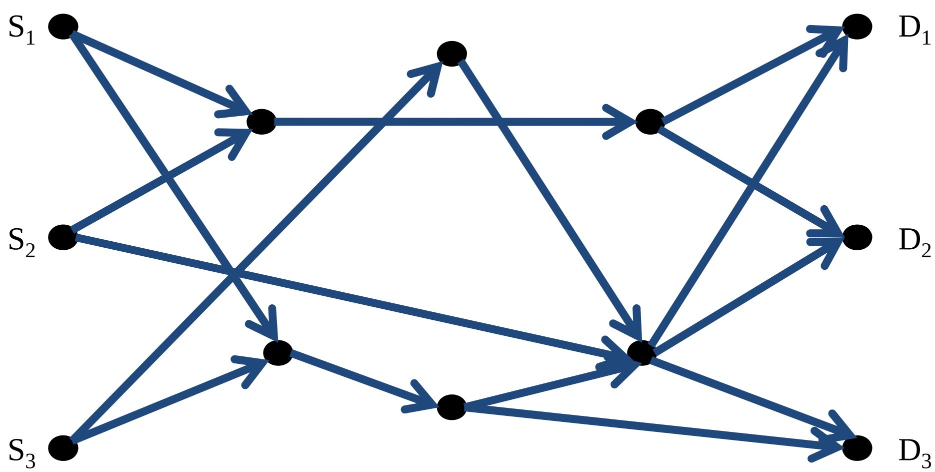
Alignment

- Interference Alignment (IA)
  - IA originally introduced for wireless interference channels
    - as a systematic way to guarantee half the rate per user
  - allows to solve fewer equations for some unknowns
    - needed when messages are mixed

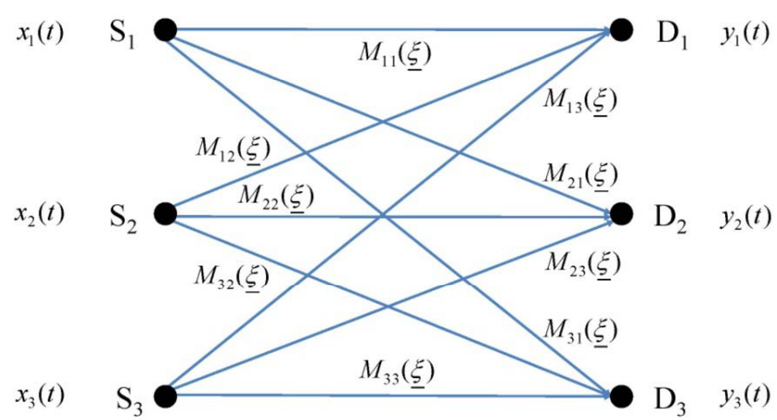
- Network Alignment (NA=NC+IA)
  - IA techniques can also be applied in networks
  - Applied to repair problem in dist. storage [WD'09, SR'10, CJM'10]
  - Applied to network coding for multiple unicasts [DVJM'10]

The Network as a Channel
Analogy

Multiple Unicast Network



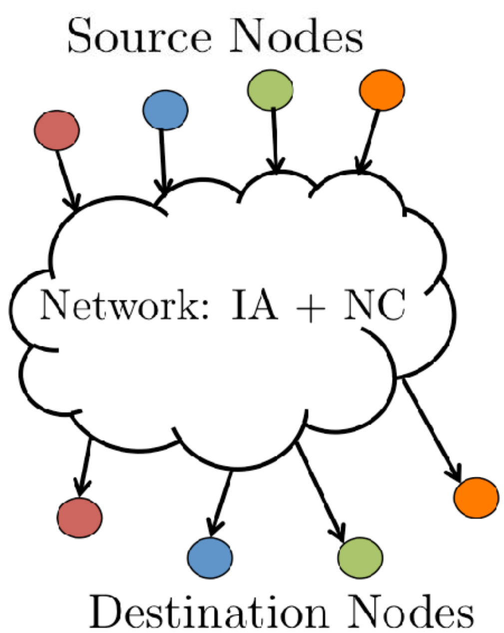
Interference Channel



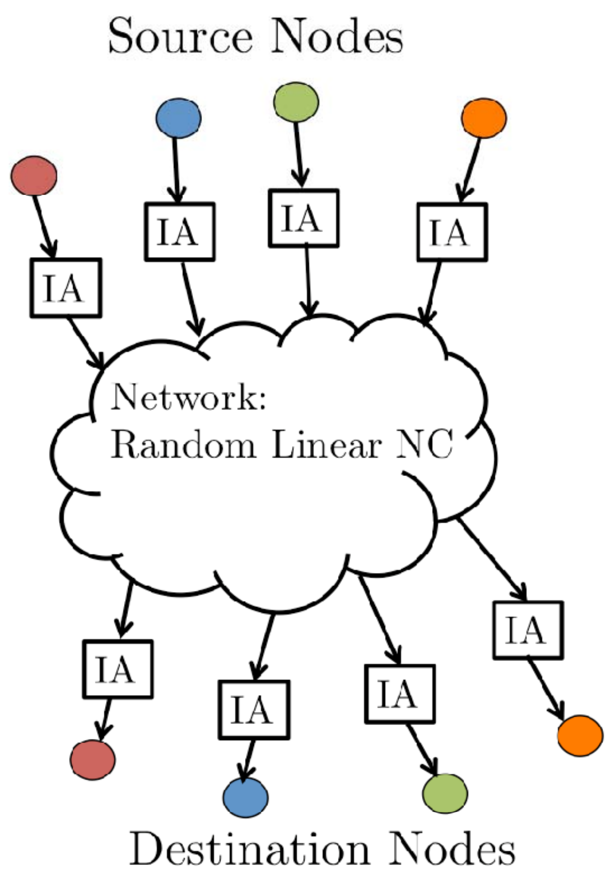
- Both represented by a Linear Transfer Function $[M_{ij}]$.
- +: $[M_{ij}]$ no longer determined by nature but defined by us
- -: Spatial dependencies introduced by the graph. Feasibility?

Network Alignment Approaches

Coding in the Middle

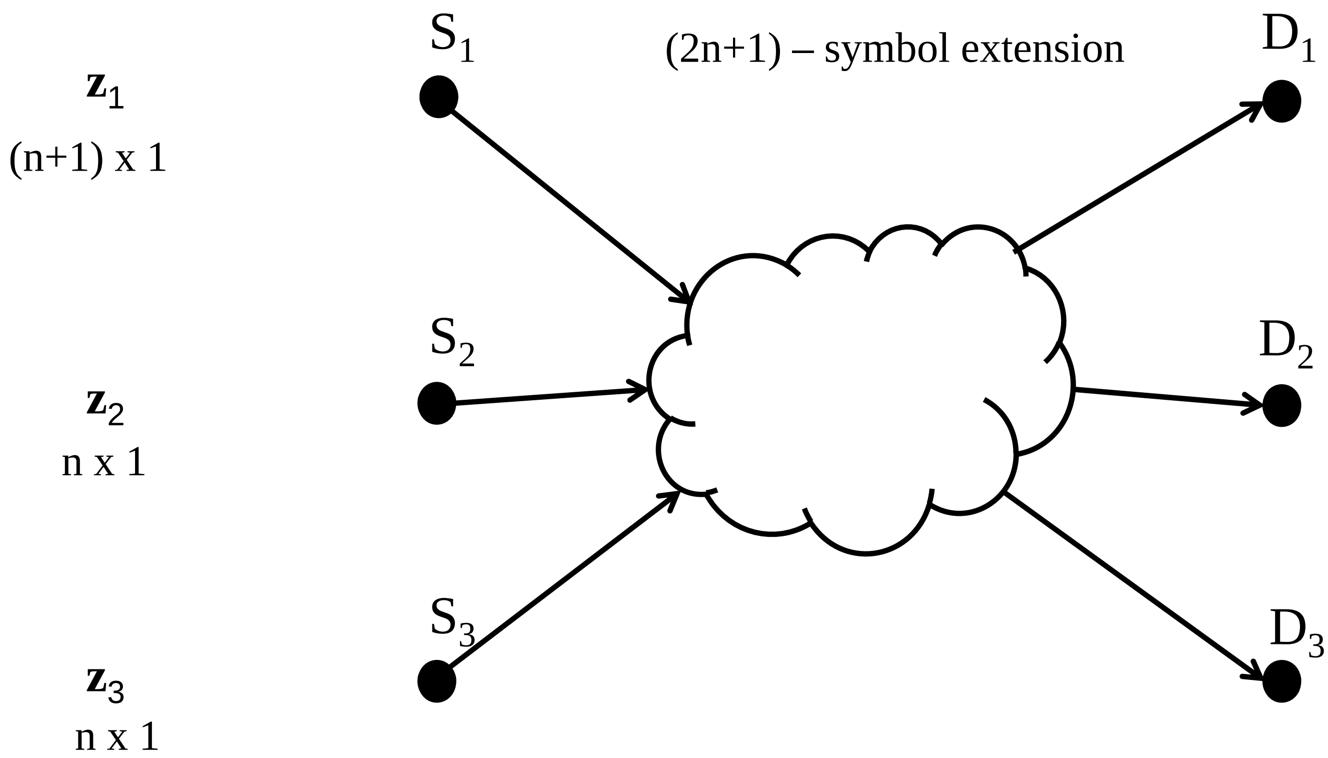


Coding at the Edge



Alignment by Coding at the Edge

- Asymptotic scheme: Symbol Extension [CJ'08]
  - recently applied to network coding for multiple unicasts [Das, Vishwanath, Jafar, Markopoulou, ISIT'10]



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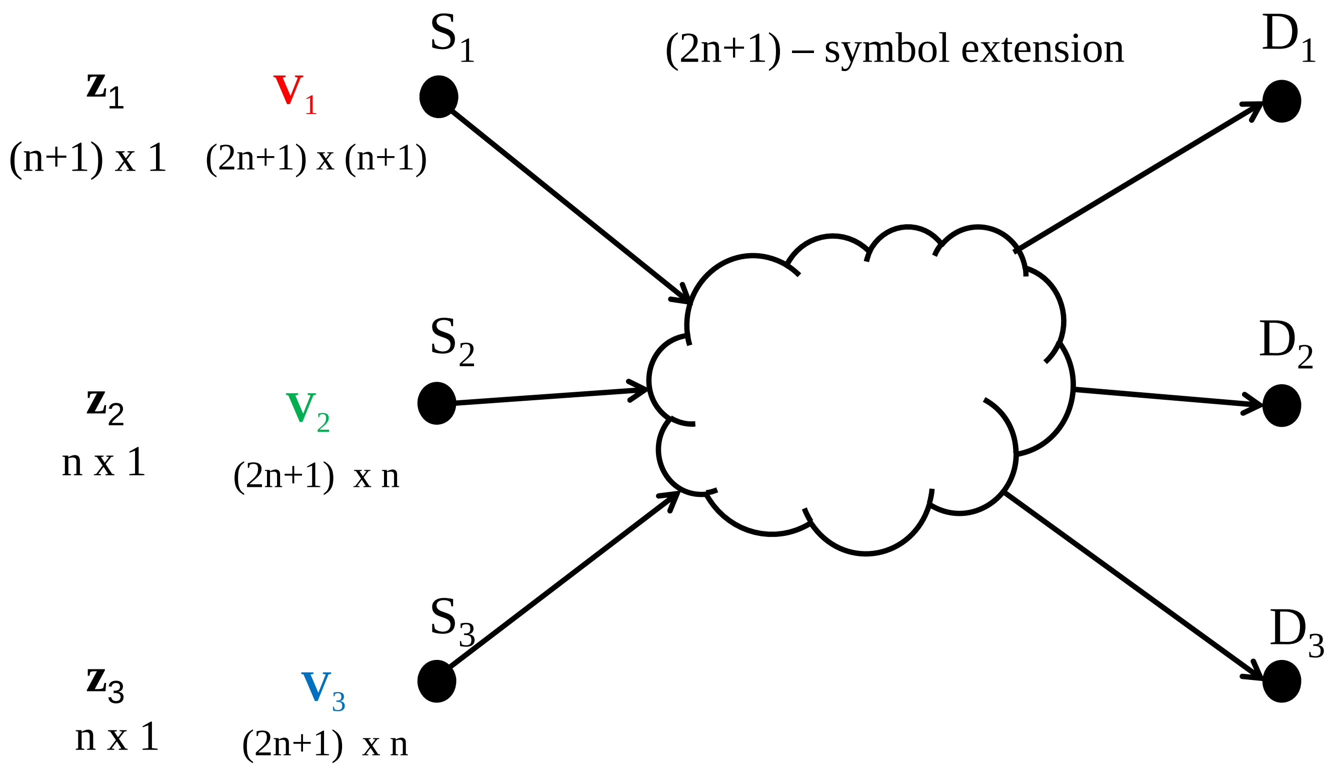

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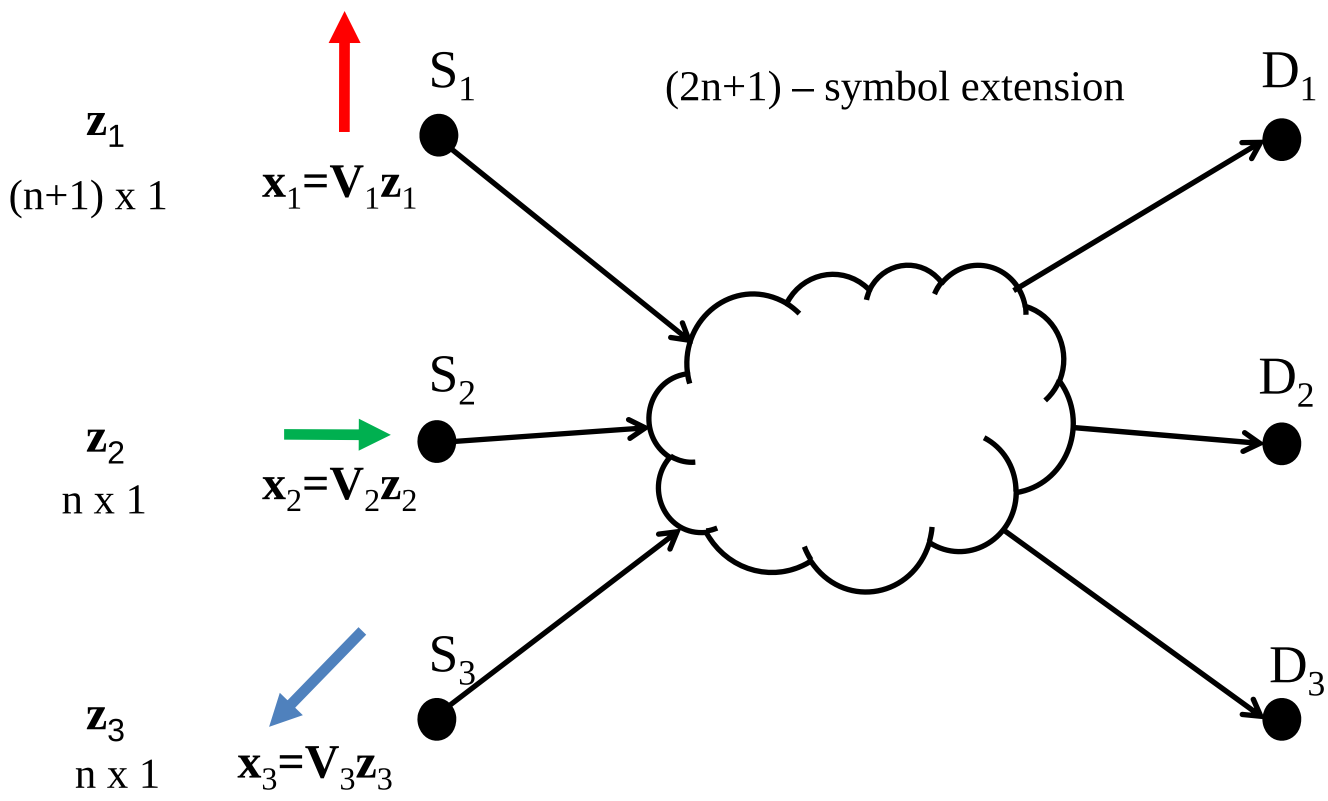

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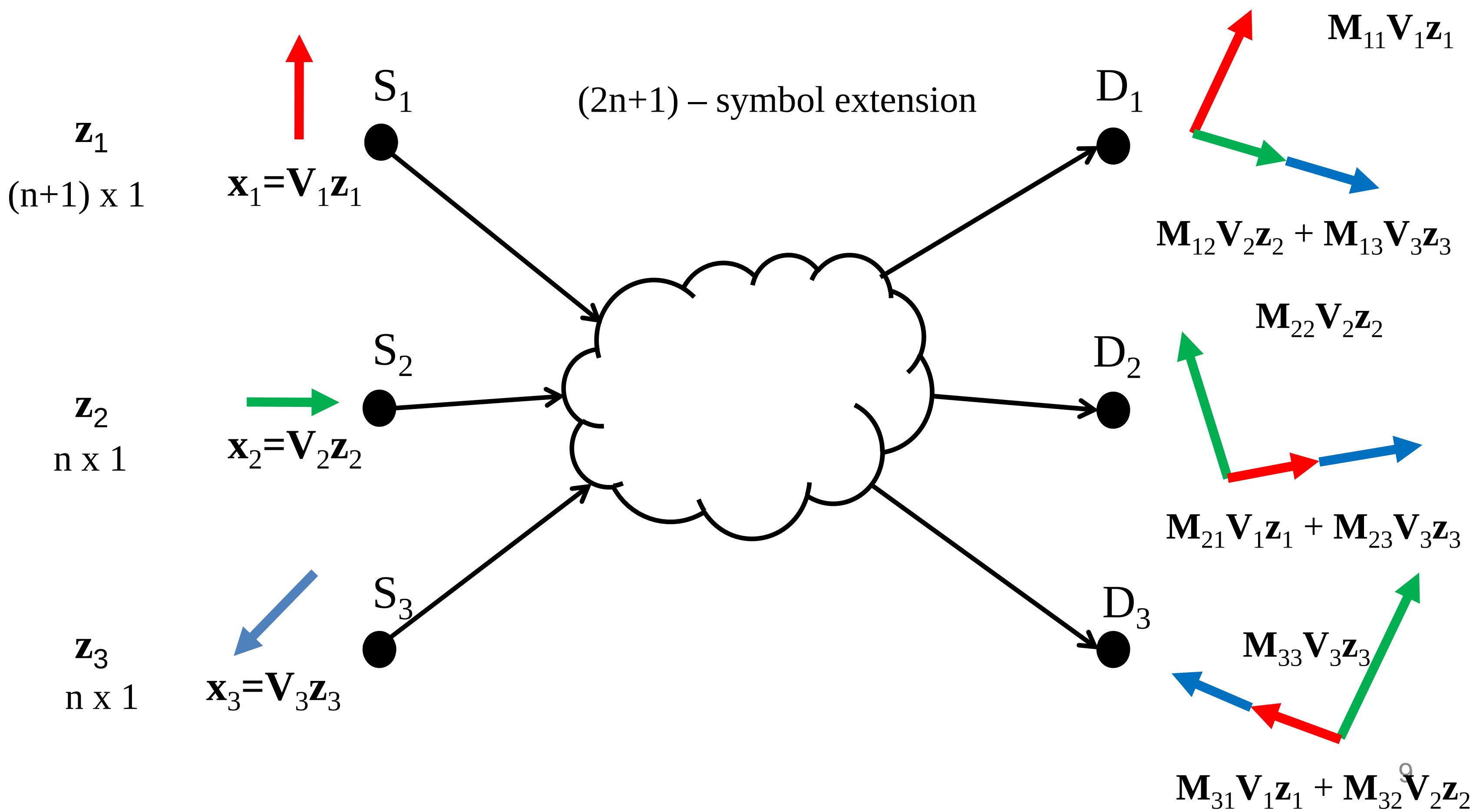

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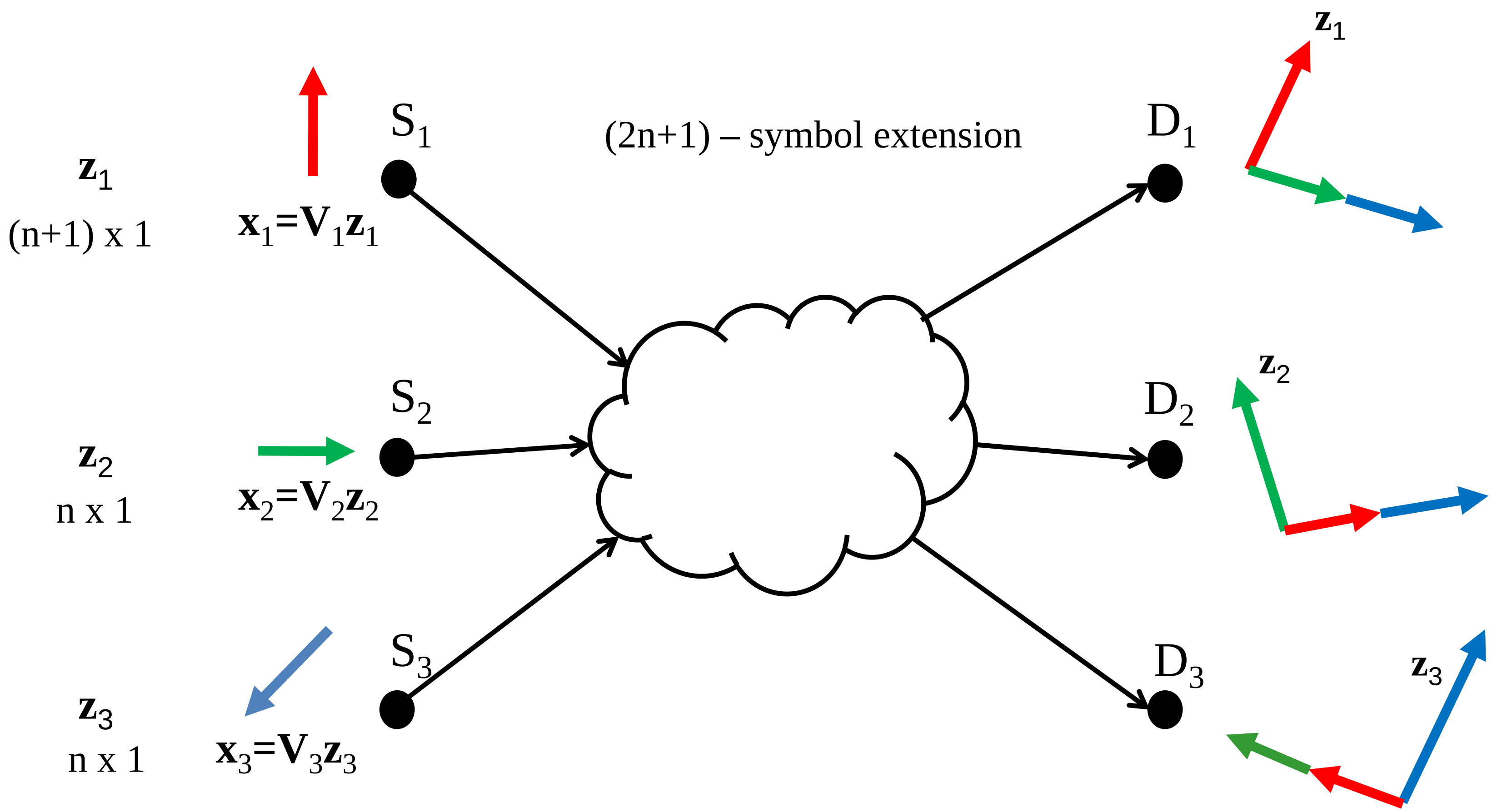


Feasibility is a challenge (unlike wireless) due to dependencies in [Mij]

Feasibility conditions
[DVJM'10]

By construction:

- $m_{ij}(\xi)$, i,j =1,2,3, non-trivial polynomials
- $m_{ii}(\xi) \neq c\, m_{ij}(\xi)$ for any c in $\mathbf{F}_p \backslash \{0\}$

Notation: $a(\xi) = m_{12}(\xi) m_{23}(\xi) m_{31}(\xi), \quad b(\xi) = m_{21}(\xi) m_{13}(\xi) m_{32}(\xi)$

Sufficient conditions for asymptotic alignment [DVJM'10] :

- for all n, and p_i, q_i (i=0,1,2,..n) it should be:

$$m_{11}(\xi) \neq \frac{m_{12}(\xi)\, m_{31}(\xi)}{m_{32}(\xi)} \frac{\sum_{i=0}^{n} p_i \left(a(\xi)/b(\xi)\right)^i}{\sum_{j=0}^{n} q_j \left(a(\xi)/b(\xi)\right)^j}$$

$$m_{22}(\xi) \neq \frac{m_{21}(\xi)\, m_{32}(\xi)}{m_{31}(\xi)} \frac{\sum_{i=0}^{n} p_i \left(a(\xi)/b(\xi)\right)^i}{\sum_{j=0}^{n} q_j \left(a(\xi)/b(\xi)\right)^j}$$

$$m_{33}(\xi) \neq \frac{m_{23}(\xi)\, m_{31}(\xi)}{m_{21}(\xi)} \frac{\sum_{i=0}^{n} p_i \left(a(\xi)/b(\xi)\right)^i}{\sum_{j=0}^{n} q_j \left(a(\xi)/b(\xi)\right)^j}$$

$\boxed{[\mathbf{M}_{11}\mathbf{V}_1 \;\; \mathbf{M}_{12}\mathbf{V}_2] \text{ is full rank}}$

Questions

When can we align:

- Intuition behind conditions? Relation to Network Structure?
- Infinitely many and complicated conditions to check. Simplify?
- How mild are these conditions? (e.g. hold almost always in wireless)

Why should we align?

- How much benefit/loss compared to alternatives?
- Is alignment necessary?

How to align?

- Is asymptotic alignment the only way? Algorithms?

Focus mostly on: 3 unicasts, min-cut =1 per session.

Understanding the conditions

- Let's look at a subset of all conditions
  - (p_0=1, q_o=1) and (p_0=1, q_1=1)

$$m_{11}(\xi) \neq \frac{m_{12}(\xi)\, m_{31}(\xi)}{m_{32}(\xi)} \qquad m_{11}(\xi) \neq \frac{m_{21}(\xi)\, m_{13}(\xi)}{m_{23}(\xi)}$$

$$m_{22}(\xi) \neq \frac{m_{21}(\xi)\, m_{32}(\xi)}{m_{31}(\xi)} \qquad m_{22}(\xi) \neq \frac{m_{12}(\xi)\, m_{23}(\xi)}{m_{13}(\xi)}$$

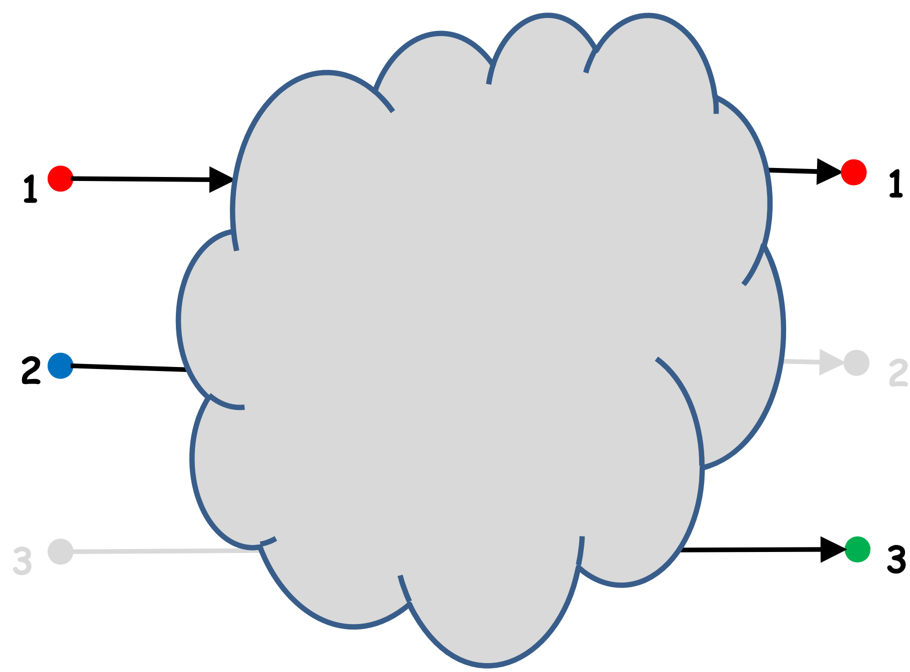
$$m_{33}(\xi) \neq \frac{m_{23}(\xi)\, m_{31}(\xi)}{m_{21}(\xi)} \qquad m_{33}(\xi) \neq \frac{m_{32}(\xi)\, m_{13}(\xi)}{m_{12}(\xi)}$$

- Interestingly, these are also necessary for any scheme to achieve rate >=1/2 in the wireless interference channel [CJ, ToIT'09]

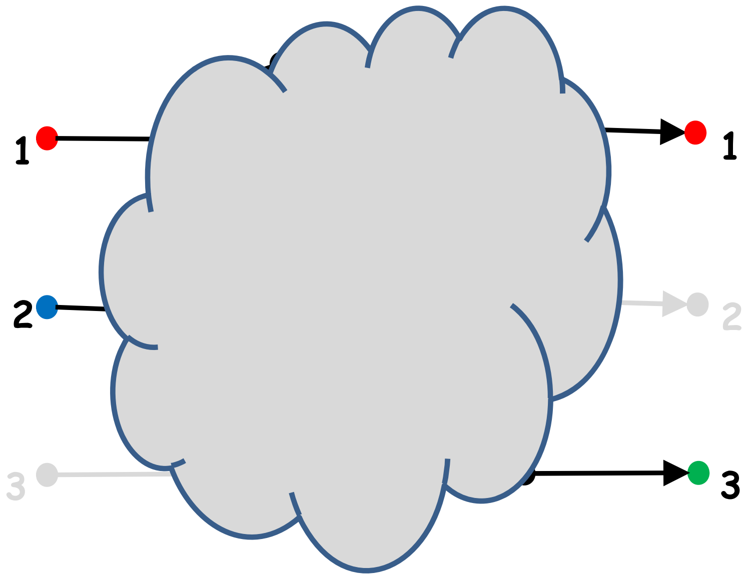
Understanding the conditions

$$\mathbf{M} = \begin{bmatrix} m_{11}(\xi) & m_{12}(\xi) & m_{13}(\xi) \\ m_{21}(\xi) & m_{22}(\xi) & m_{23}(\xi) \\ m_{31}(\xi) & m_{32}(\xi) & m_{33}(\xi) \end{bmatrix}$$

$$m_{11}(\xi) \neq \frac{m_{12}(\xi)\, m_{31}(\xi)}{m_{32}(\xi)}$$



Conditions and network structure



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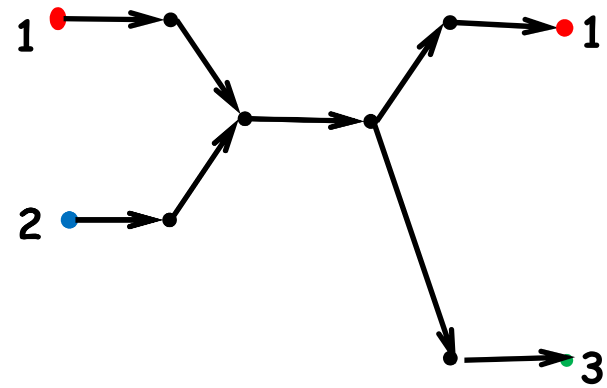

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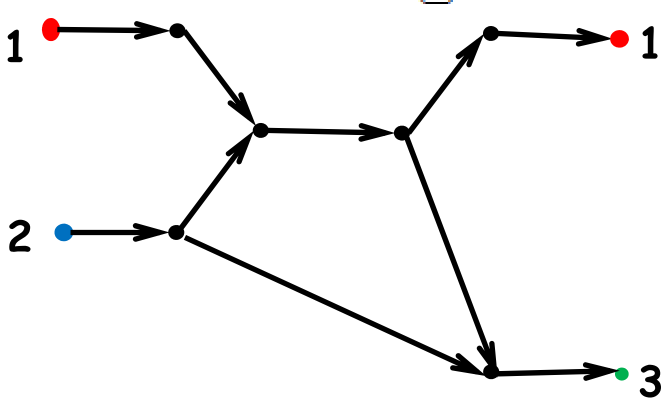

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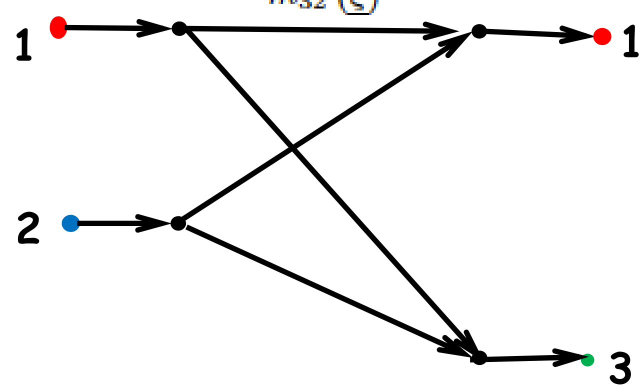

Simplifying the conditions?

- <u>Conjecture:</u>
  The "small" conditions are sufficient for asymptotic alignment.

$$m_{11}(\xi) \neq \frac{m_{12}(\xi)\, m_{31}(\xi)}{m_{32}(\xi)} \qquad m_{11}(\xi) \neq \frac{m_{21}(\xi)\, m_{13}(\xi)}{m_{23}(\xi)}$$

$$m_{22}(\xi) \neq \frac{m_{21}(\xi)\, m_{32}(\xi)}{m_{31}(\xi)} \qquad m_{22}(\xi) \neq \frac{m_{12}(\xi)\, m_{23}(\xi)}{m_{13}(\xi)}$$

$$m_{33}(\xi) \neq \frac{m_{23}(\xi)\, m_{31}(\xi)}{m_{21}(\xi)} \qquad m_{33}(\xi) \neq \frac{m_{32}(\xi)\, m_{13}(\xi)}{m_{12}(\xi)}$$

$$\overset{?}{\Longrightarrow}$$

$$m_{11}(\xi) \neq \frac{m_{12}(\xi)\, m_{31}(\xi)}{m_{32}(\xi)} \frac{\sum_{i=0}^{n} p_i \left(a(\xi)/b(\xi)\right)^i}{\sum_{i=0}^{n} q_j \left(a(\xi)/b(\xi)\right)^j}$$

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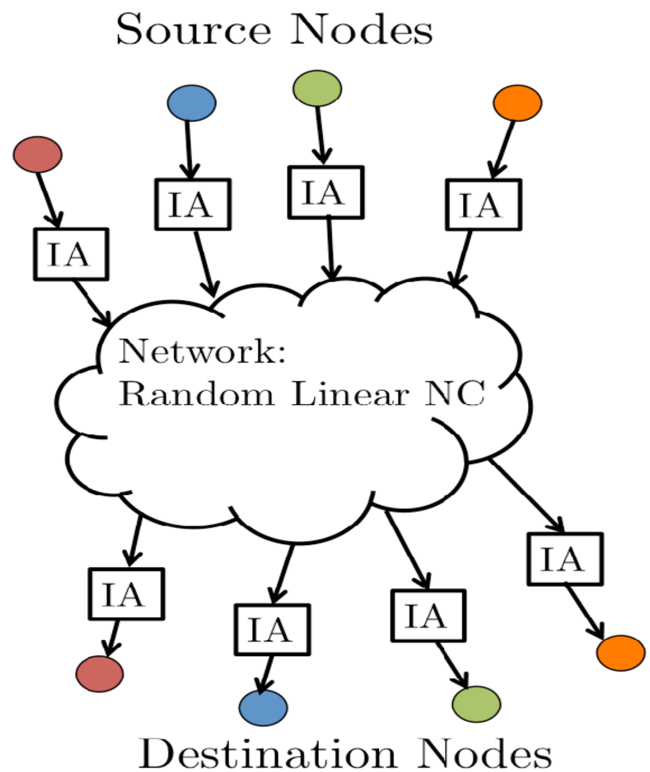
Alignment Approaches

to Code at the Edge or in the Middle?

Coding at the Edge



<u>Symbol Extension Method:</u>

- intelligence at the edge, middle is simple
- widely applicable
- large number of symbols and finite field

Example of Coding in the Middle

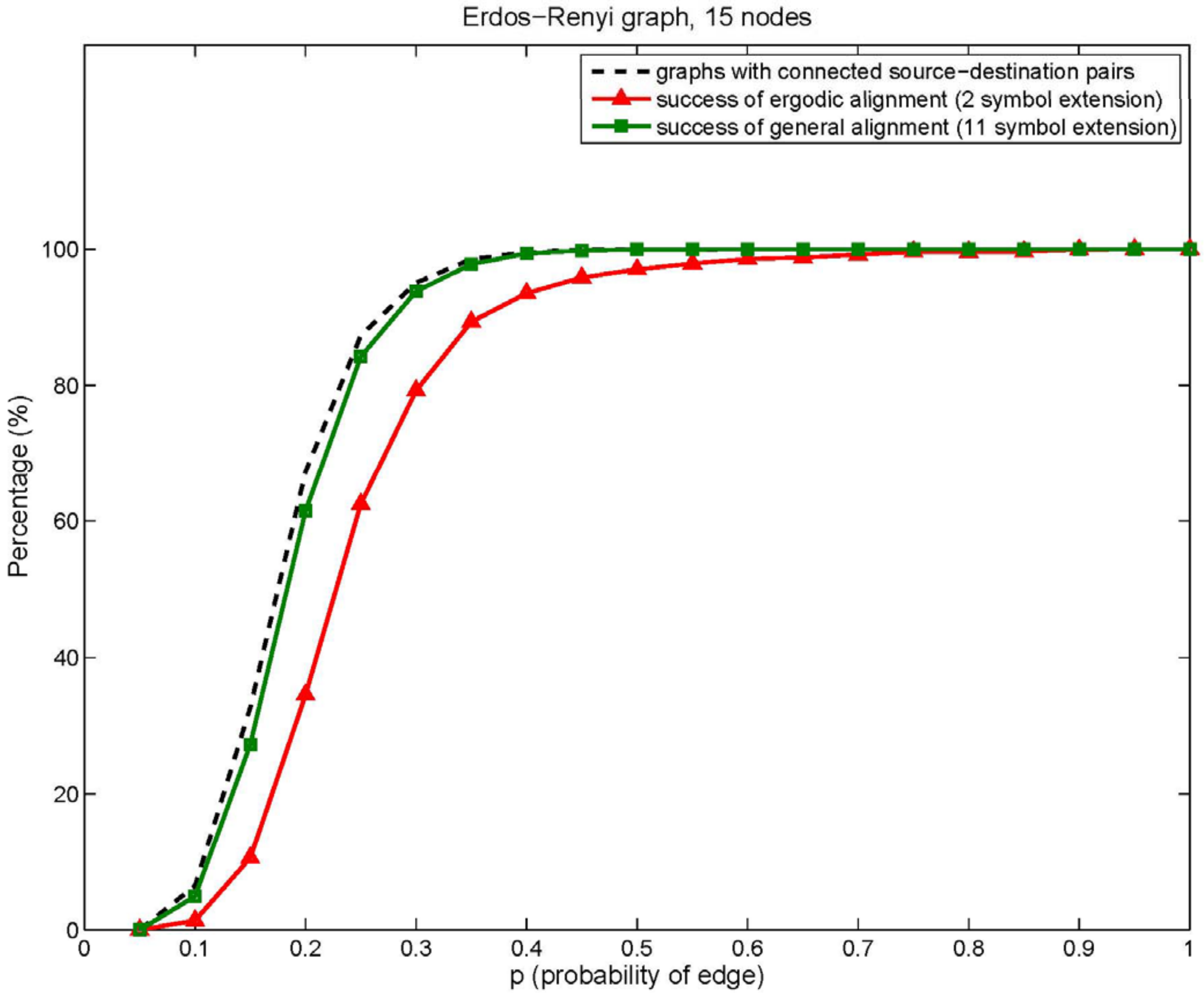
- "Ergodic" Alignment
  - inspired by [Nazer, Gastpar, Jafar, Vishwanath, "Ergodic IA", ISIT'09]
  - choose coding coefficients over two time slots so that:

$$M^{(1)} = \begin{bmatrix} m_{11}^{(1)} & m_{12} & m_{13} \\ m_{21} & m_{22}^{(1)} & m_{23} \\ m_{31} & m_{32} & m_{33}^{(1)} \end{bmatrix} \qquad M^{(2)} = \begin{bmatrix} m_{11}^{(2)} & m_{12} & m_{13} \\ m_{21} & m_{22}^{(2)} & m_{23} \\ m_{31} & m_{32} & m_{33}^{(2)} \end{bmatrix}$$

  - then subtract and obtain 1 symbol in 2 time slots
- Feasibility condition:
  - **each m_{ii} is not a function of the m_{ij}'s**
  - more restrictive than the condition for asymptotic alignment
- Strengths:
  - 2 time slots are enough (no need to wait for channel opportunities)
  - smaller field size and number of symbols than asymptotic scheme
- Limitation:
  - Intelligence resides in the network. May be complex for large networks.

How mild are the conditions?

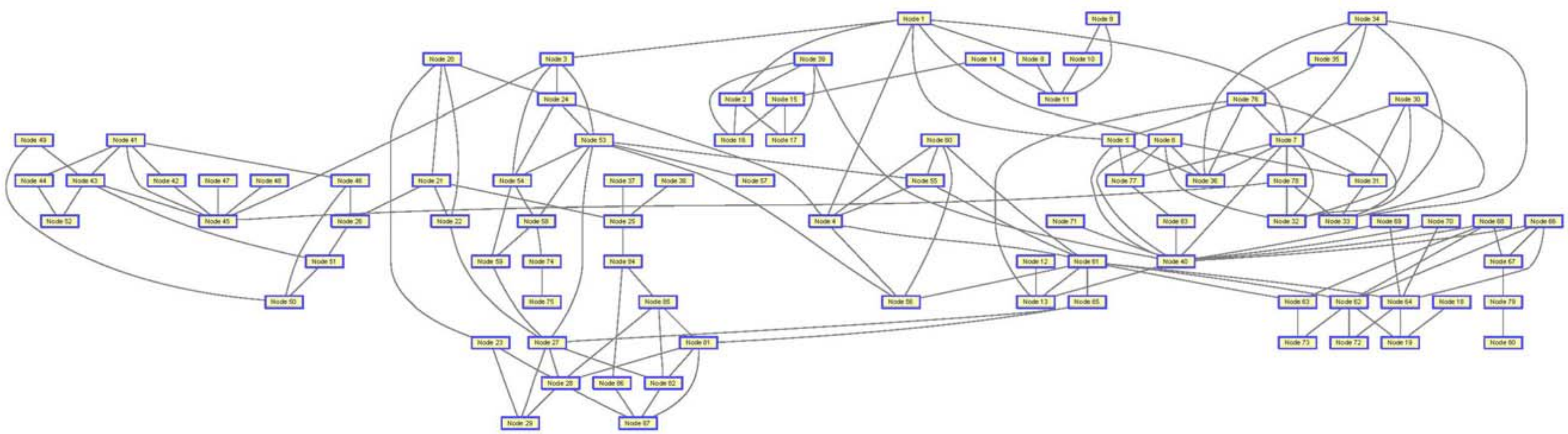
Random Graphs



How mild are the conditions?

Real Topologies

Network	Number of nodes	Number of edges
ASN-1221: Telstra (AUS)	108	153
ASN-1239: Sprintlink (USA)	315	972
ASN-1755: EBONE (EU)	87	161
ASN-3257:Tiscali (EU)	161	328
ASN-3967:Exodus (USA)	79	147
ASN-6461:Abovenet (USA)	141	374
ABILENE	11	14
BELNET	15	27
GEANT	23	37



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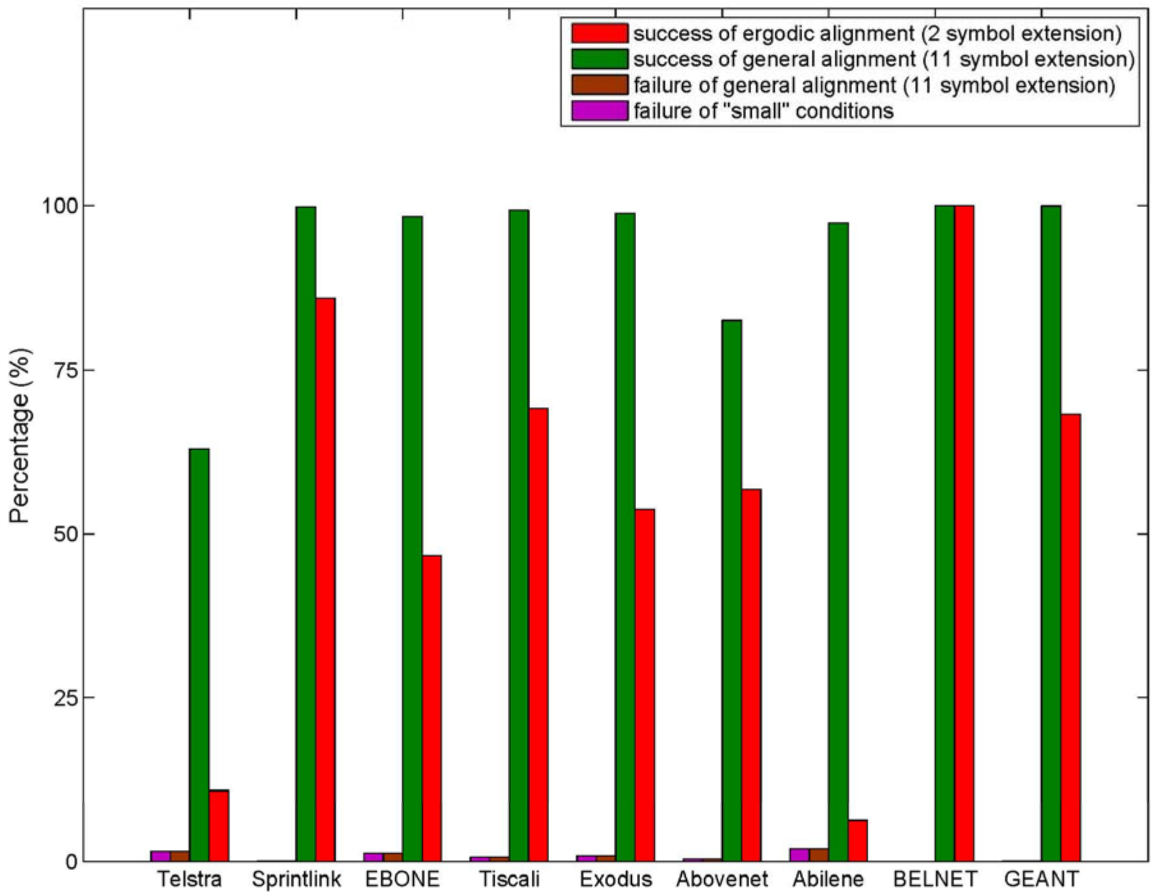
Real Topologies



Outline

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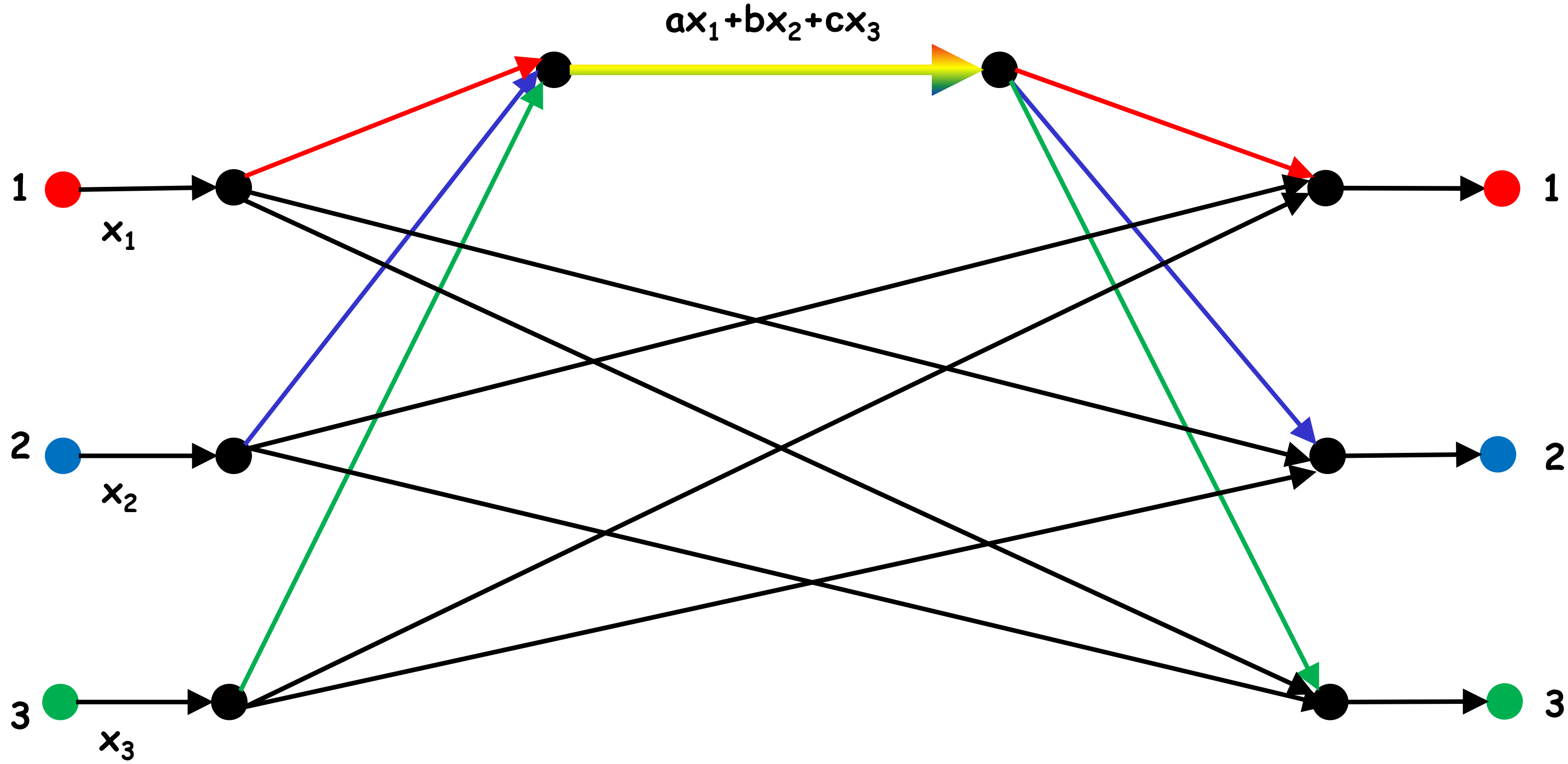
Example: Extended Butterfly

- K unicast sessions going through the same bottleneck
- Routing achieves rate = 1/k
- Network coding (with all side links) achieves rate = 1
- Alignment (with sufficiently rich side links) achieves rate =1/2

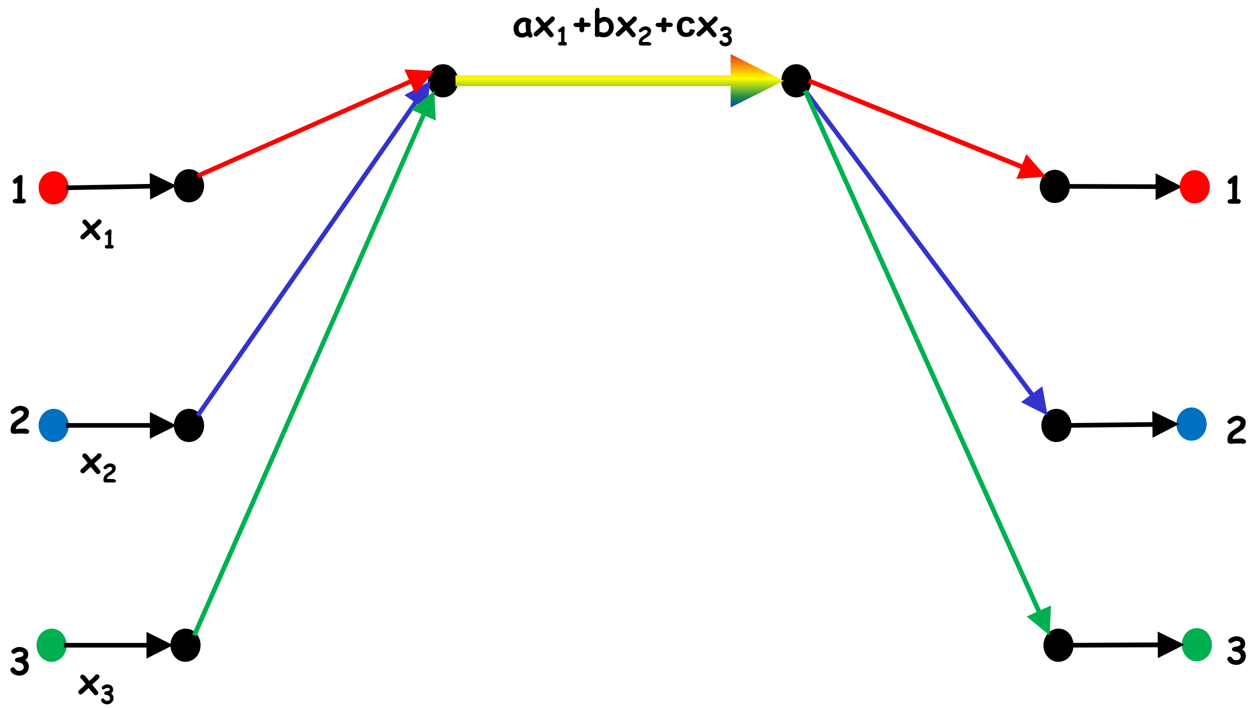


Example 1

3 unicast sessions, min-cut=1

- No side links
- Rate 1/3 by any scheme

$$M = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

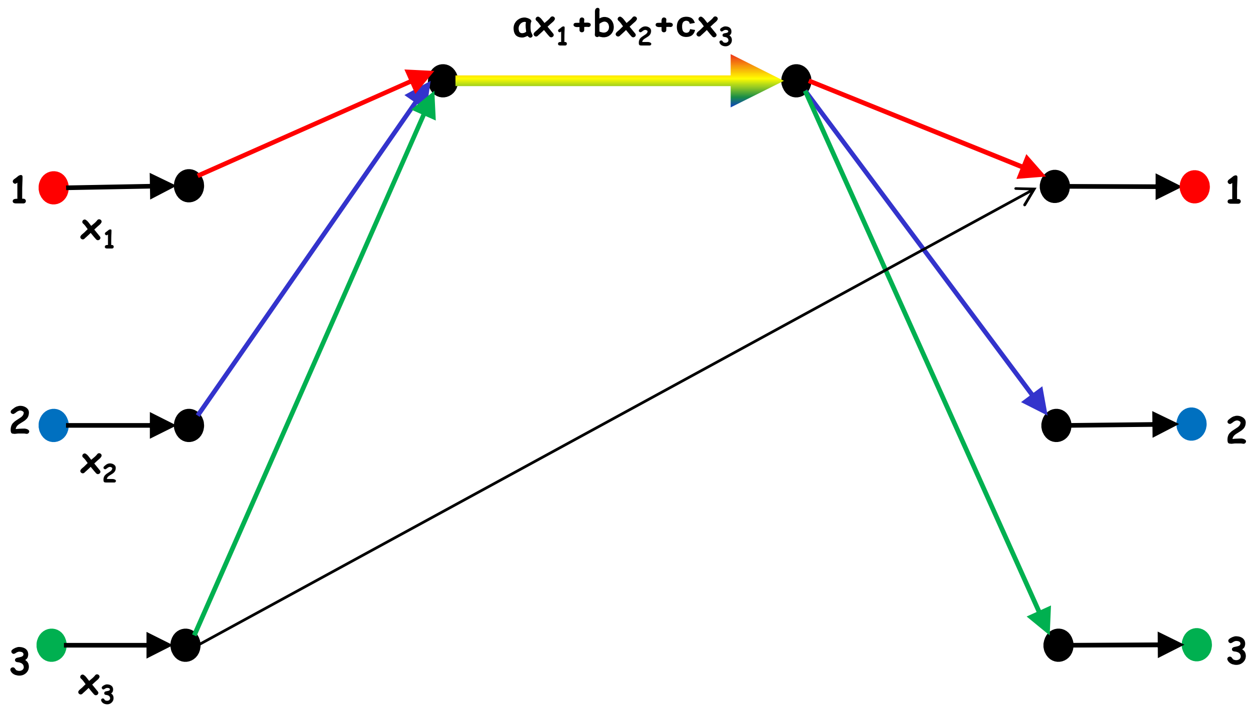


Example 1

3 unicast sessions, min-cut=1

- One side link
- Still rate 1/3 by any scheme

$$M = \begin{bmatrix} a & b & c+c_{13} \\ a & b & c \\ a & b & c \end{bmatrix}$$

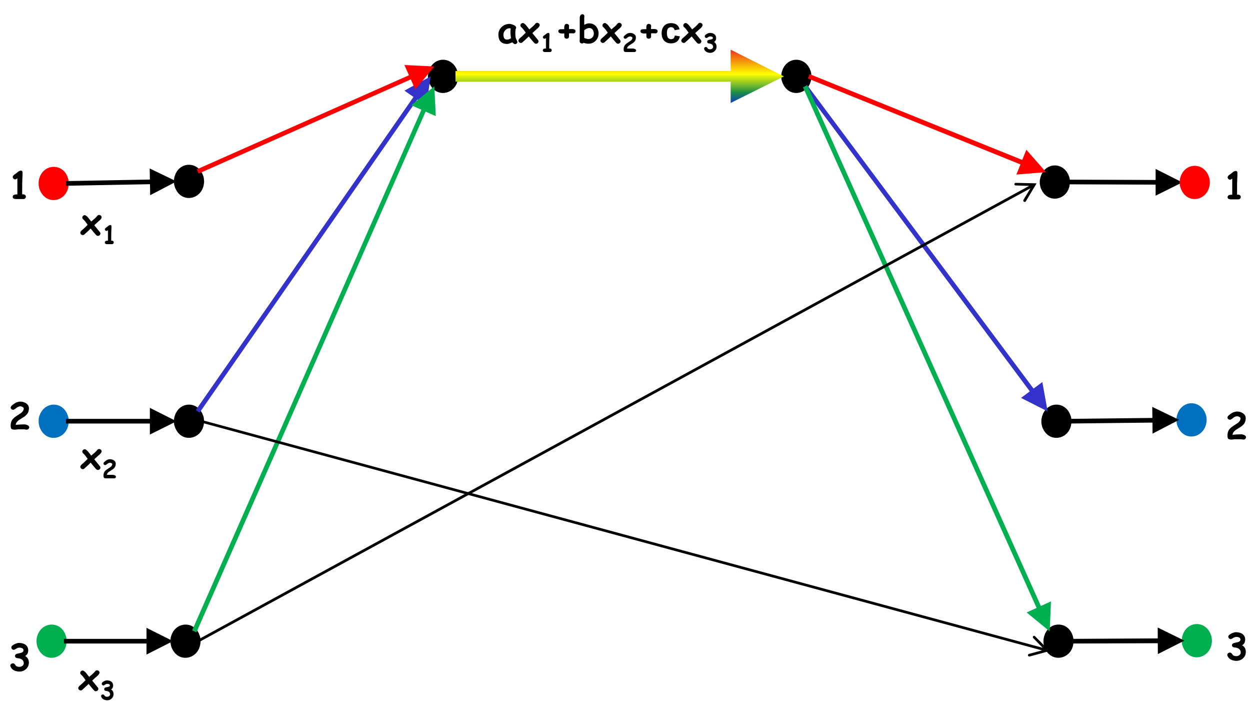


Example 1

3 unicast sessions, min-cut=1

- Two receivers have side links (which ones matter)
- Here: still 1/3 rate, NA not possible

$$M = \begin{bmatrix} a & b & c + c_{13} \\ a & b & c \\ a & b + c_{32} & c \end{bmatrix}$$

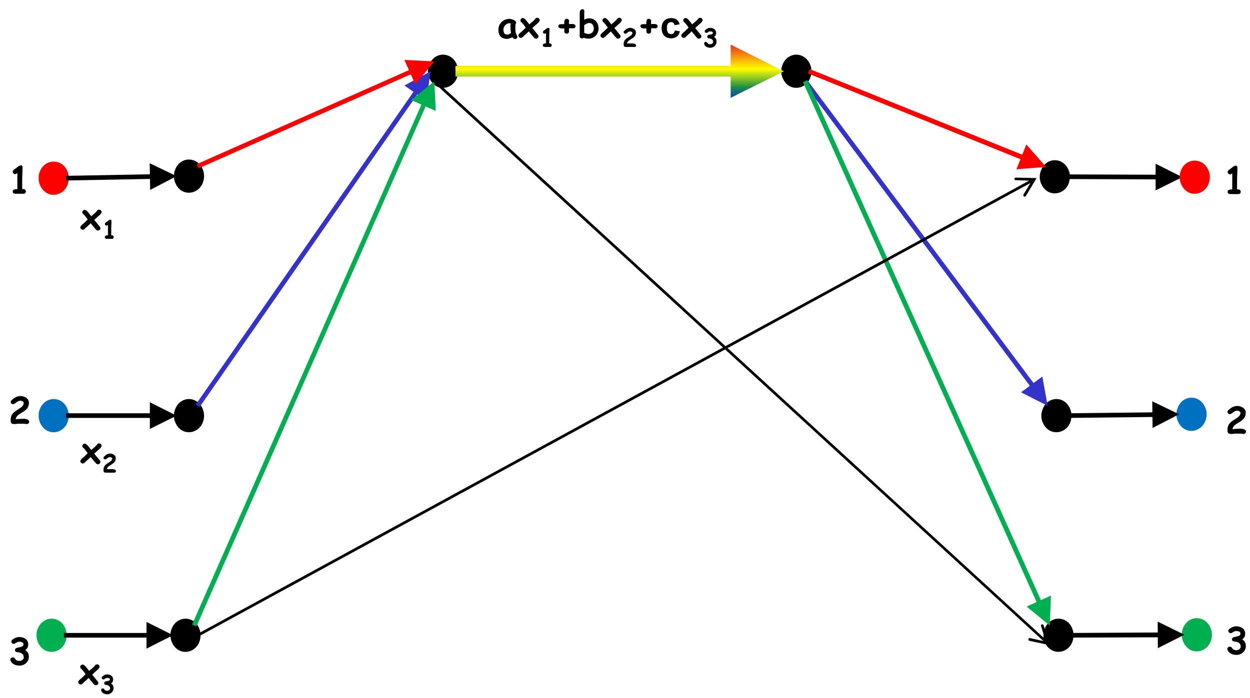


Example 1

3 unicast sessions, min-cut=1

- Two receivers have side links
- 1/2 min-cut achievable in 2 slots
  - by alignment or
  - or by sharing between session 2 and butterfly 1-3

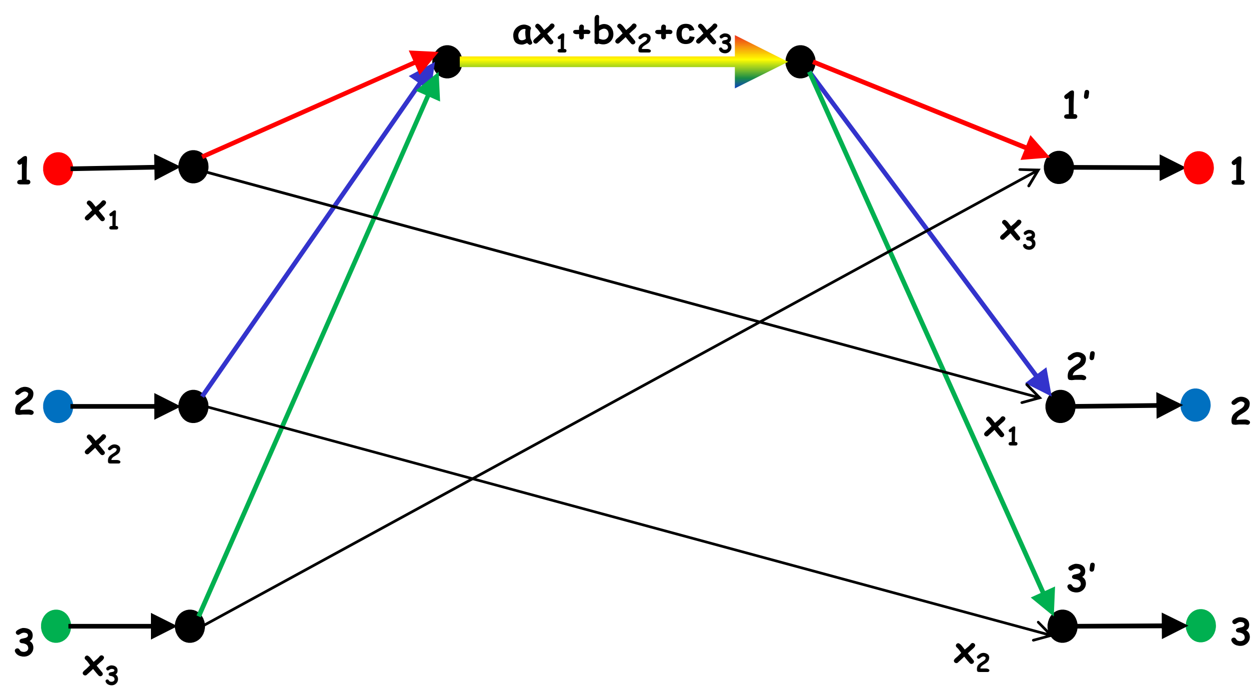
$$M = \begin{bmatrix} a & b & c + c_{13} \\ a & b & c \\ a + c_{31} & b & c \end{bmatrix}$$



Example 1

3 unicast sessions, min-cut=1

- Three receivers have one side link.
- The optimal rate is ½ the min-cut,
- NA achieves it; routing does not; there are no butterflies
- ***Alignment needed*** if intelligence is allowed only at the edge
- Coding in the middle (RNC + deterministic at 1',2',3') can also achieve ½ the min-cut



When is NA necessary?

summary

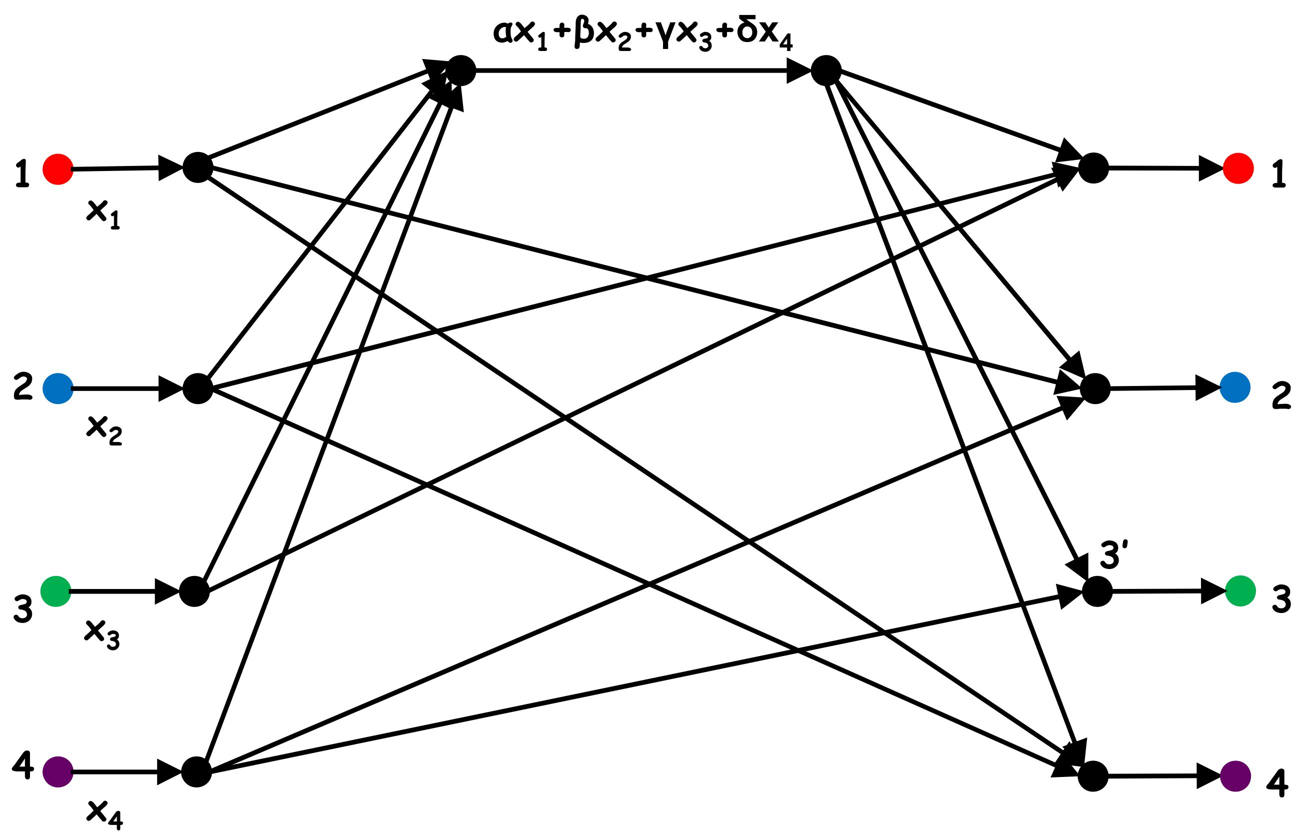
- Arbitrary network, 3 unicast sessions with min-cut=1:
  - Theorem: Whenever NA is possible, another approach (e.g., routing, butterflies, NC in the middle w/o alignment) can also achieve half the min-cut.
  - Proof outline:
    - Sparsity bound S=1/3: the extended butterfly examples extend to any network
    - Sparsity bound >=1/2: consider networks where routing rate <= 1/2, and construct a deterministic NC scheme (in the middle, w/o alignment) that achieves ½ the min-cut

- NA necessary (depending on the topology) to achieve ½ for:
  - K>3 unicasts
  - or min-cut>1

Example 2:

4 unicast sessions, min-cut=1

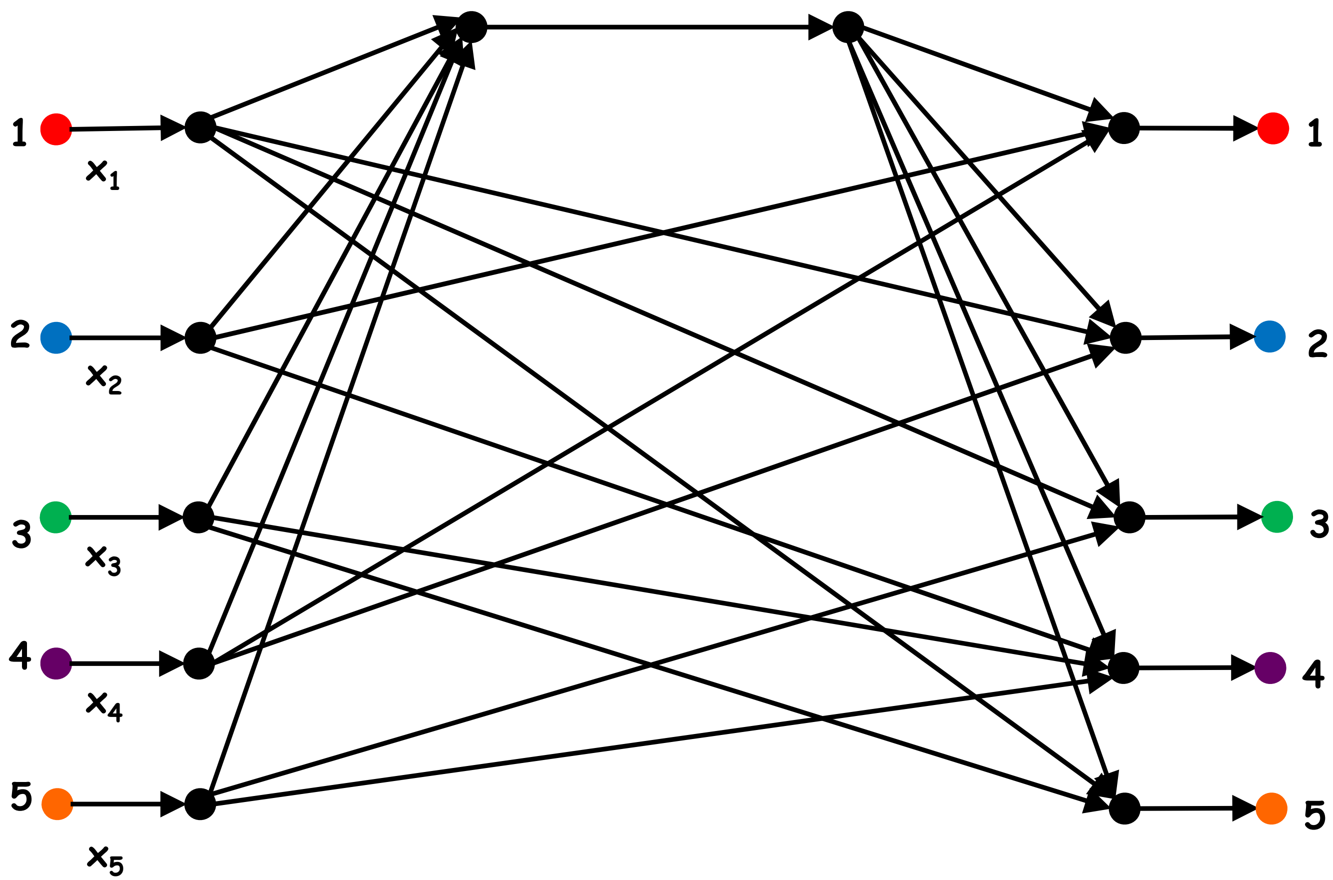
- **Alignment needed:**
  - receiver 3 has 2 equations, 4 unknowns in 2 slots
  - even node 3′ has 3 equations, 4 unknowns in 2 slots
  - alignment achieves ½ min-cut, no other scheme does



Example 2:

5 unicast sessions, min-cut=1

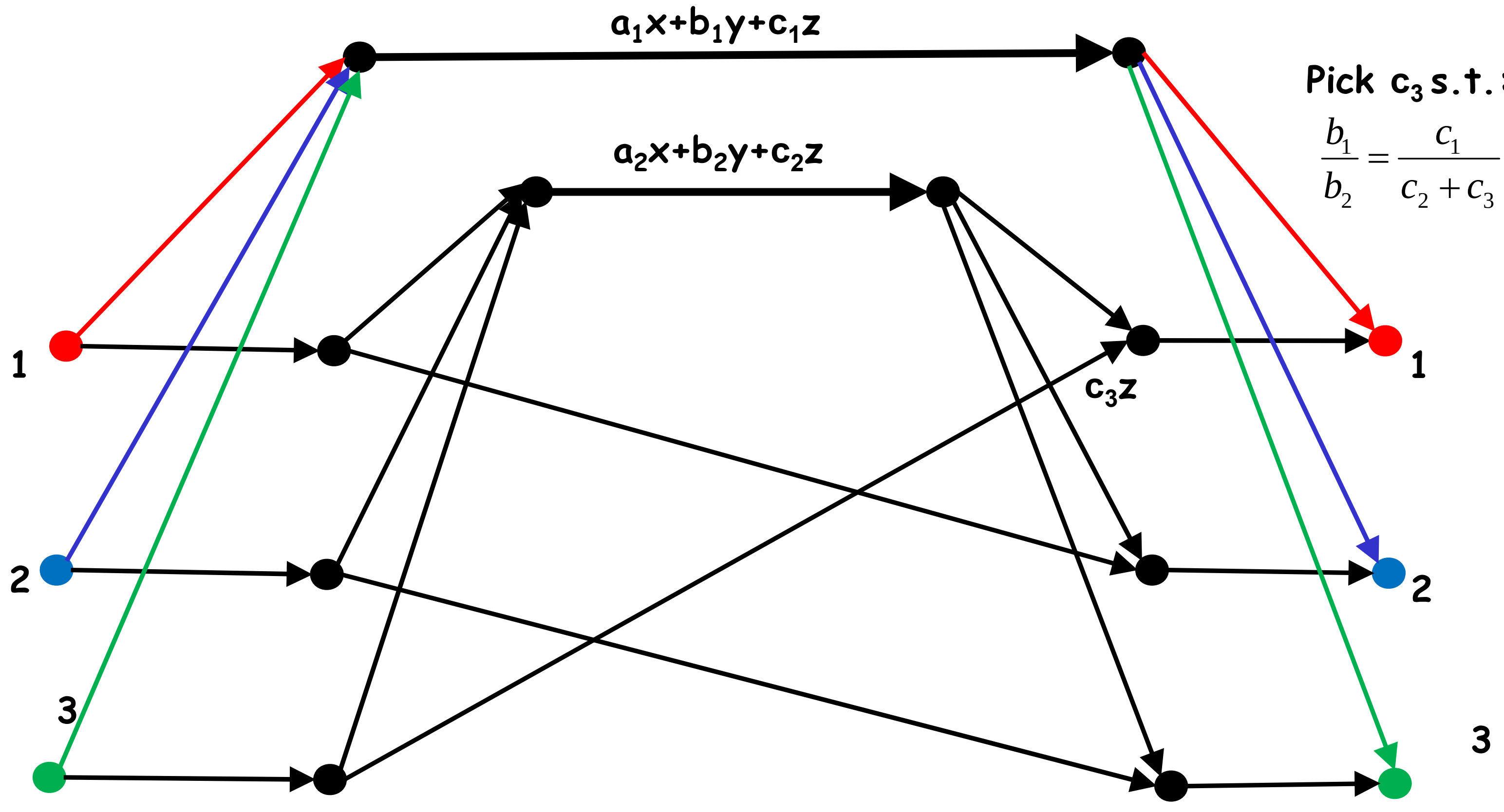
- **Alignment needed** - effect amplified
- all receivers have 2 (nodes in the middle have 2) equation,s with 5 unknowns in 2 slots
- alignment achieves $\frac{1}{2}$ min-cut, no other scheme does



Example 3:

3 unicasts sessions, min-cut = 2

- **Alignment needed:**
  - It achieves $\frac{1}{2}$min-cut, which is optimal; no other scheme achieves $\frac{1}{2}$ rate.



Pick c_3 s.t.:

$$\frac{b_1}{b_2} = \frac{c_1}{c_2 + c_3}$$

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  - Can other schemes achieve $\frac{1}{2}$ the min-cut or more?
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Conclusion

- Network Alignment (NA=NC+IA)
  - a systematic approach for network coding across multiple unicasts
  - guarantees half the min-cut per session

- Future Directions:
  - Characterize Feasibility and Performance of NA and their relation to Network Structure
  - Develop practical Alignment Algorithms.