On Signal Processing Methods for MIMO Relay Architectures

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Abstract—Relay networks have received considerable attention recently, especially when limited size and power resources impose constraints on the number of antennas within a wireless sensor network. In this context, signal processing techniques play a fundamental role, and optimality within a given relay architecture can be achieved under several design criteria. In this paper, we extend recent optimal minimum-mean-square-error (MMSE) and SNR designs of relay networks to the corresponding multiple-input-multiple-output (MIMO) scenarios, whereby the source, relays and destination comprise multiple antennas. We shall investigate maximum SNR solutions subject to power constraints and zero-forcing (ZF) criteria, as well as approximate MMSE equalizers with specified target SNR and global power constraint.

I. INTRODUCTION

The interest in relay networks has been recently increased from several different perspectives, all of which range from the modeling of link abstractions at higher layers in a communication system, to coding, synchronization, and signal processing designs within a physical layer. One of the basic motivations for the use of cooperative communications lies in the exploitation of the spatial diversity provided by network nodes, as well as on the efficient use of power resources, which can be achieved by a scheme that simply receives and forwards a given information under a certain optimality criterion. A recent review on several aspects of cooperative communications can be found, for instance in [1], and in the references therein.

In addition to the several information-theoretic approaches on relay networks [2], architectures based on an amplify-and-forward schemes have been recently analyzed more rigorously within signal processing frameworks [3],[4]. More specifically, while [2] provides capacity scaling laws for certain MIMO relay protocols, [3] studies optimal power distribution designs under MMSE and SNR criteria, focusing on a SISO relay scheme, and [4] targets maximizing mutual information between source and destination under joint power distribution at the source and relay, considering a single relay structure constituted by multiple antennas.

It has been shown that the use of MIMO wireless networks significantly improves spectral efficiency and link reliability through spatial multiplexing and space-time coding respectively. In this paper, we extend prior work to encompass the MIMO case, where each relay is itself equipped with multiple receive and transmit antennas, and intra-relay cooperation is allowed. In all schemes, we assume that the source has no information on the channels, which are perfectly known at the relay and the destination, and spatially multiplexes data (i.e. transmitting statistically independent data streams from different antennas). Also the source distributes power equally on each transmit antenna. In the following section, we define the problem formulation and the design criteria to be investigated further ahead.

Notation: We shall use lower case letters for vectors, while capital letters represent matrices. The complex transposition operator is defined as \(^\ast\), while the conjugate of the elements of a matrix \(A\), is given by \(\bar{A}\). Also the \(\text{tr}(A)\) is the trace of matrix \(A\). The operation \(\text{vec}(A)\) stacks the columns of \(A\) into a single column vector, and \(\otimes\) denotes the kronecker product. We denote by \((A_0 \oplus A_1 \oplus \cdots)\) a block diagonal matrix with block elements given by \(A_i\). Also, \((\cdot)^\dagger\) denotes the pseudo-inverse.

II. PROBLEM FORMULATION

Figure 1 illustrates a wireless sensor network consisting of \(M\) transmit antennas at the source and \(M\) receive antennas at the destination. We consider an amplify-and-forward relay scheme consisting of \(K\) relays, each relay equipped with \(N\) receive and transmit antennas. The relay matrix is represented by a block diagonal matrix \(F\), where each block is given by a \(N \times N\) relay gain matrix \(F_i\), \(i = 0, 1, \ldots, K - 1\). We denote by \(H_s = [H_{s0} \ldots H_{sK-1}]^\dagger\) the \(KN \times M\) channel matrix between the source and the relay nodes, while \(H_t = [H_{t0} \ldots H_{tK-1}]\) is the \(M \times KN\) channel matrix between the relay sensors and the destination. The channel matrices are memoryless and a quasi-static fading condition is assumed. The received signal can be modeled as

\[
y_t = H_t F H_s s + H_t F v_s + v_t
\]

where \(v_s\) and \(v_t\) are additive Gaussian noise (AGN) with covariance matrices \(R_{v_s}\) and \(R_{v_t}\) respectively and \(s\) is the transmitted signal with covariance matrix \(R_s = \text{Es}s^\ast\). A two-phase (two-hop) protocol is used to transmit data from the source to the receiver. In the first phase (hop) the source broadcasts a signal vector \(s\) towards the relay sensors. In the second phase, the relay sensors retransmit the information to the destination. We further assume synchronous transmission and reception at relays nodes. At the receiver, depending on the design criterion, one can further employ a MIMO equalizer, which we shall denote later by \(K\), in order to
compensate for the effect of the overall MIMO channel. In
the latter, the goal is to design \( \{ F, K \} \) both separately and
jointly, considering global power constraint.

![MIMO Relay Architecture with intra-relay cooperation.](image)

In the following, we investigate the performance of a MIMO
relay network under two optimization criteria:

1) First, without considering any predetermined MIMO de-
coder (equalizer), we design the optimal relay matrix \( F \)
such that it maximizes the output SNR without post-
equalization (see \( y_t \), in Fig. 1). This is achieved under
a zero-forcing (ZF) criterion, first with a specified target
SNR. Then, by relaxing the target SNR condition,
a total power constraint for the relays is enforced.
Note that in such cases, the role of the relays is to provide
equalization for the underlying forward and backward
channels.

2) Second, we design the relay matrix and equalizer pair
\( \{ F, K \} \) under a MMSE criterion, considering two
distinct cases: (i) without power constraint, where \( \{ F, K \} \)
are designed in two independent steps; (ii) under a global
power constraint, where the relays are selected such that
the overall MMSE is minimized.

III. SNR APPROACH

In [3], the authors have provided, for the special case of
single-input-single-output (SISO) antennas scheme, optimal
relay coefficients that maximize the signal power at the
receiver input, subject to both local and global power constraints.
In particular, SNR maximization subject to local power
constraints leads to the traditional power normalization and phase
compensation method employed in the SISO amplify-and-
forward scheme. For the global power optimization, the au-
thors provide an approximate expression for the relay gains.
In this paper, we find exact expressions for optimal relay
matrices in the MIMO case. In the approach considered herein,
the two noise sources appearing in Fig. 1 are taken into
consideration, even though the SNR can be defined in different
ways, depending on the optimality criteria at hand. In this
section, we define the output SNR as the ratio between the
power of the input signal and the overall contribution of the
noise sources, whose effect is transferred to the output node.
Thus, without any equalization criteria enforced, one may seek
the optimal relay matrix \( F \) that solves

\[
\max_F \frac{\text{tr}(H_j F H_i R_{\delta} H_s^* F^* H_t^*)}{\text{tr}(H_j F R_{\delta} F^* H_t^*) + \text{tr}(R_{\delta})} \tag{2}
\]

This problem can be shown to be equivalent to the well
known generalized eigenvalue problem (without any con-
straint), stated in the form of a Rayleigh-Ritz ratio. Note that
(2) can be expressed as

\[
\max_F \frac{\text{vec}(H_j^* F^* H_t^*) \text{vec}(H_s^* F^* H_t^*)}{\text{vec}(F^* H_t^*) \text{vec}(F^* H_s^*) + \text{tr}(R_{\delta})} \tag{3}
\]

Now, since \( F \) is block diagonal, we have that

\[
\text{vec}(H_j^* F^* H_t^*) = \sum_{i=0}^{K-1} \text{vec}(H_j^* F_t^* H_{i}^*)
\]

\[
= \sum_{i=0}^{K-1} (\tilde{H}_{t_0} \otimes H_{s_0}^*) \text{vec}(F_t^*)
\]

\[
= \left[ \begin{array}{c} 
(\tilde{H}_{t_0} \otimes H_{s_0}^*) \cdots (\tilde{H}_{t_{K-1}} \otimes H_{s_{K-1}}^*) \end{array} \right] f, \tag{5}
\]

where \( f \triangleq [f_0^* \ f_1^* \ \cdots \ f_{K-1}^*] \), and \( G \) defined in (5) has
dimension \( M^2 \times KN^2 \). In the same fashion, we have that

\[
\text{vec}(F^* H_t^*) = \mathcal{P} \left[ \begin{array}{c} 
\text{vec}(F_{t_0}^* H_{t_0}^*) \\
\vdots \\
\text{vec}(F_{t_{K-1}}^* H_{t_{K-1}}^*) 
\end{array} \right]
\]

\[
= \mathcal{P} \left[ \begin{array}{c} 
(\tilde{H}_{t_0} \otimes I_N) \oplus \cdots \oplus (\tilde{H}_{t_{K-1}} \otimes I_N) \end{array} \right] f, \tag{6}
\]

where \( \mathcal{P} \) is simply a permutation matrix that reorganizes the
entries of \( E f \) accordingly, and where we have further defined
\( E \triangleq \left[ \begin{array}{c} 
(\tilde{H}_{t_0} \otimes I_N) \oplus \cdots \oplus (\tilde{H}_{t_{K-1}} \otimes I_N) \end{array} \right] \). Thus, the
Rayleigh-Ritz ratio in (3) can be expressed as

\[
\max_f \frac{f^* \mathcal{G}^* (I_M \otimes R_{\delta}) \mathcal{G} f}{f^* [E^* \mathcal{P}^* (I_M \otimes R_{\delta}) \mathcal{P} E] f + \text{tr}(R_{\delta})}
\]

\[
= \max_f \frac{f^* A f}{f^* B f + M\sigma_{\delta}^2}, \tag{7}
\]

where in the latter, we have defined

\[
B \triangleq E^* \mathcal{P}^* (I_M \otimes R_{\delta}) \mathcal{P} E, \tag{8}
\]

and used the fact that \( \text{tr}(R_{\delta}) = M\sigma_{\delta}^2 \).

A. Maximum Output SNR subject to Zero-Forcing Constraint

Under a ZF constraint, we are required to solve (2) subject
to \( \tilde{H}_t F H_s = \eta I \). The gain \( \eta \) can be defined, for instance,
based on the desired target SNR, as

\[
\eta \triangleq \sqrt{\text{SNR}_{\delta}(\sigma_{\delta}^2/\sigma_s^2)} \tag{9}
\]
where \( \sigma_f^2 \) and \( \sigma_r^2 \) are signal and noise powers at the receiver. In this case, the numerator in (7) becomes \( \eta^2 \text{tr}(R_s) \). Moreover, using (5) and defining \( b \triangleq \text{vec}(I) \), our problem is now

\[
\max_f \frac{\sigma_f^2 |\text{tr}(\eta I)|^2}{f^*Bf + M\sigma_f^2} \text{ subject to } Gf = \eta b, \tag{10}
\]

where \( B \) incorporates the effect of noise at the relays. Note that in order for \( f \) to have a solution, \( G \) must at least be a square matrix, which requires that \( M \leq \lceil \sqrt{KN} \rceil \), where \([\cdot]\) denotes truncation. Thus, assume \( B \) is positive definite and consider the spectral factorization \( B = Q \Delta Q^* \). Defining \( \bar{f} = \Delta^{1/2}Q^*f \), (10) can be expressed as

\[
\max_{\bar{f}} \frac{\sigma_f^2 M^2 \eta^2}{||\bar{f}||^2 + M\sigma_f^2} \text{ subject to } (GQ\Delta^{-1/2}) \bar{f} = \eta b. \tag{11}
\]

The solution to this problem is the minimum norm vector \([5]\) \( \bar{f} \) that satisfies the linear constraint above, which is given by \( \bar{f} = \eta(GQ\Delta^{-1/2})b \). This results in

\[
f = \eta B^{-1}G^*(GB^{-1}G^*)^{-1} \text{vec}(I), \tag{12}
\]

where \( B \) is defined in (8). Note that \( E^*P^*(I_M \otimes R_v)PE \) is positive definite if \((I_M \otimes R_v)P\) is positive definite, which requires that the \( KN \times KN \) matrix \( E \) is full column rank. This implies that \( N \leq M \leq \lceil \sqrt{KN} \rceil \). Now, we may remark that, in case \( M < N \), \( B \) becomes non-negative definite, and the above expression no longer holds. However, consider instead the spectral decomposition

\[
B = Q \begin{bmatrix} \Delta & 0 \\ 0 & I \end{bmatrix} Q^*, \tag{13}
\]

and define \( z \triangleq Q^*f = \begin{bmatrix} Q_1^*f \\ Z_2 \end{bmatrix} \). Because we have freedom to choose \( z_1 \), many solutions exist. Moreover, one possible solution is obtained by setting \( z_1 = 0 \), so that it corresponds to a minimum norm vector \( \bar{f} \). In this case, the problem becomes

\[
\min_{\bar{z}} \bar{z}^* \Delta \bar{z} \quad \text{s.t.} \quad GQ^*\bar{z} = \eta b, \tag{14}
\]

and the solution in this case can be verified to have a form similar to (12).

**B. Maximum Output SNR Subject to Global Power Constraint**

When a global power constraint is enforced, it may not be possible to achieve a predefined target SNR. That is, let \( p \) be the total power to be distributed among the relays. This implies that

\[
\sum_{i=0}^{K-1} \text{tr}(F_i R_{r_i} F_i^*) = f^*Wf = p, \tag{15}
\]

where \( R_{r_i} = H_{t_i} R_s H_{t_i}^* + R_{v_0} \), \( i = 0, 1, \ldots, K - 1 \), represents the input signal powers to each individual relay, and

\[
W = [(I_N \otimes R_{v_0}) \oplus (I_N \otimes R_{r_1}) \oplus \cdots \oplus (I_N \otimes R_{r_{K-1}})] \tag{16}
\]

represents the weighting for the norm of \( f \).

Now, if the minimum norm solution in (12) is such that (15) is not satisfied, one may need to readjust the target SNR in order to meet the power specification. That is, the power constrained solution is given by (12), where \( \eta \) is

\[
\eta = \frac{\sqrt{p}}{\sqrt{b^* (GB^{-1}G^*)^{-1} GB^{-1} W B^{-1} G^* (GB^{-1}G^*)^{-1} b}}. \]

### IV. MMSE Approach

In this section, we shall first consider an MMSE design given a certain target SNR. One possible approach is to proceed in two steps, where first we design \( F \), which, along with \( H_t \), equalizes for the effect of \( H_s \) and the noise \( v_t \). Then, given \( F \) we proceed by equalizing for the overall channel and the output noise \( v_t \). This approach has been considered in [3] for a single antenna scheme.

![MIMO Relay Equalization](image_url)

**A. MMSE with Target SNR — Two-step Equalization**

Let us define

\[
\bar{F} \triangleq H_t F = \begin{bmatrix} H_{t_0} F_{0} & H_{t_1} F_{1} & \cdots & H_{t_{K-1}} F_{K-1} \end{bmatrix}, \tag{17}
\]

and consider the receiver target SNR defined by the constant \( \eta \) in (9). Thus, the MMSE solution \( \bar{F} \) to

\[
\min_{\bar{F}} \mathbb{E} ||\eta s - \bar{F} y_s||^2 \tag{18}
\]

is given by

\[
\bar{F} = \eta R_s H_s^* (R_{v_0} + H_s R_s H_s^*)^{-1} = \eta R_{v_0}^{-1} + H_s^* R_{v_0}^{-1} H_s^{-1} H_s^* R_{v_0}^{-1}. \tag{19}
\]

We assume that \( N \geq M \), so that the system of equations (17) always has a solution. The choice of a particular solution can be accommodated depending on some extra criteria. For instance, since one is normally interested in a low power consumption by the relay sensors, we can pick \( \{F_i\} \) as the one with minimum norm, i.e., \( F_i = H_t^* F_i \), or

\[
F_i = \eta H_{t_i}^*(H_{t_i} H_{t_i}^*)^{-1} (R_{v_0}^{-1} + H_s^* R_{v_0}^{-1} H_s^{-1} H_s^* R_{v_0}^{-1}) \tag{20}
\]

Thus, given \( \bar{F} \), we can now easily express the MMSE equalizer \( K \) as

\[
K = (R_{v_0}^{-1} + H_s^* F_s^* H_s^* R_{v_0}^{-1} H_s^* F_s^*)^{-1} H_s^* F_s^* H_s^* R_{v_0}^{-1}. \tag{21}
\]

Where we have defined the effective noise variance due to \( v_s \) at the output of \( H_t \) as

\[
R_{v_0} \Delta \triangleq H_t F R_{v_0} F^* H_t^* + R_{v_0}. \tag{22}
\]
where each $V_i$ corresponds to the $i$-th $(M \times N)$ block element of $V$. Now substituting (29) into (15) gives

$$\alpha = \frac{\sqrt{p}}{\sqrt{\sum_{i=0}^{K-1} \text{tr} \left[ R_{v_i}^{-1/2} (H_s H_s^*)^{-1} R_{v_i}^{1/2} V_i V_i^* \right]}}.$$ 

V. SIMULATION RESULTS

We assume that all relays are at equal distance from the source and destination so that the forward and backward channels have the same statistics, which in turn are generated as zero-mean and unit-variance independent and identically distributed (i.i.d) complex Gaussian random variables. Also, independent QPSK symbols are transmitted through each antenna with total power $M$ uniformly distributed among the antennas. The noise variances are assumed to be the same for all antennas. We plot symbol error rate (SER) curves versus SNR, which is defined per bit per antenna at each relay antenna.

Figure 3 shows the SER for ZF and MMSE in two steps for the case $M = 3$, $N = 3$, for $K = 1, 2$ and $5$. We remark that for the sake of comparison, we have chosen $M = N$, since for the ZF criterion it is required that $M \geq N$, while for the MMSE, we must satisfy $M \leq N$. Increasing the number of relays improves system performance, and we may note that both MMSE and ZF criteria behave similarly. As Fig. 3 shows at $\text{SER}=10^{-3}$ we obtain 10 dB gain when we increase the number of relays from 1 to 2. This gain is mainly because of the distributed diversity gain that we get by adding one more relay which has, in this case, 3 antennas. It should be mentioned that since we have not imposed any power constraint at this point, the ZF and MMSE criteria choose the best power in order to maximize the SNR or minimize MMSE at the input to the receiver. Considering the same setting as in Fig. 3, we then constrain the total output relay power to 10 dB, which is illustrated in Fig. 4. It can be verified that the MMSE criterion performs better as we increase the number of relays. In this case we find optimally the matrix $F$ from (21), and we further adjust the total output power to $p$. Hence, the solution to the maximization problem in (26) becomes

$$\hat{F} = \alpha V^*.$$ 

Let us consider that the transmit symbols and noise at the relays are white, i.e., $R_s = \sigma_s^2 I$ and $R_{v_i} = \sigma_{v_i}^2 I$ respectively, then

$$\left( R_s + \frac{1}{\alpha^2} I \right) -\sigma_s^2 H_s^* H_s = \left( \sigma_{v_i}^2 H_s^* H_s + \frac{1}{\alpha^2} I \right) -\sigma_{v_i}^2 H_s^* H_s = V^* \sigma_{v_i}^2 \left( \sigma_s^2 \Delta + \left( \frac{1}{\alpha^2} + \sigma_{v_i}^2 I \right) \right)^{-1} \Delta V,$$

where

$$H_s^* \Delta = \begin{bmatrix} V^* & V_1 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & V_1 \end{bmatrix}.$$ 

$F_i = \alpha H_{t_i}^* (H_{t_i} H_{t_i}^*)^{-1} R_{v_i}^{1/2} V_i$, (29)

where each $V_i$ corresponds to the $i$-th $(M \times N)$ block element of $V$. Now substituting (29) into (15) gives

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where

$$H_s^* \Delta = \begin{bmatrix} V^* & V_1 \end{bmatrix} \begin{bmatrix} \Delta & 0 \\ 0 & V_1 \end{bmatrix}.$$ 

$F_i = \alpha H_{t_i}^* (H_{t_i} H_{t_i}^*)^{-1} R_{v_i}^{1/2} V_i$, (29)
VI. CONCLUSIONS

In this paper, we have proposed MIMO amplify-and-forward relaying strategies designed based on commonly used equalization criteria. We have investigated the performance under ZF and MMSE criteria with and without power constraint. Without power constraint, relays can have different total output power in order to achieve a desired Quality of Service (QoS) at the destination. Also, the choice of \( N \) and \( M \) defines the type of criterion to be used (\( M \leq N \) for the MMSE, or \( N \leq M \leq \sqrt{KN} \) in the ZF case). For the same number of antennas at the source and relays, the MMSE solution outperforms the ZF solution under power constraint. The former, however, requires extra complexity, due to a MIMO equalizer at the receiver. In a future work, we shall elaborate on further designs on optimal power distribution as well as minimum bit error rate performance criteria.

![Fig. 3. SER performance of Maximum Output SNR s.t. ZF and MMSE in two Steps for different number of relays.](image1)

![Fig. 4. SER performance of Joint MMSE and ZF s.t. power constraint (PC) of 10 dB for different number of relays.](image2)

![Fig. 5. SER performance of ZF and Joint MMSE s.t. power constraint (PC) for one channel realization and total output power ranging from 0 dB to 20 dB. In this figure, \( M = 3 \), \( N = 3 \), and \( K = 5 \).](image3)

![Fig. 6. SER performance of ZF and MMSE in two Steps for fixed number of relays.](image4)

REFERENCES


